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EFFECTS OF CHROMATICITY AND SYNCHROTRON EMISSION ON COUPLED-BUNCH TRANSVERSE STABILITY*

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Abstract

We present a theory that can compute the transverse coupled-bunch instability growth rates at any chromaticity and for any longitudinal potential provided only that the long-range wakefield varies slowly over the bunch. The theory is expressed in terms of the usual coupled-bunch eigenvalues at zero chromaticity, and when the longitudinal motion is simple harmonic our solution only requires numerical root-finding that is easy to implement and fast to solve; the more general case requires some additional calculations, but is still relatively fast. The theory predicts that the coupled-bunch growth rates can be significantly reduced when the chromatic betatron tune spread is larger than the coupled-bunch growth rate at zero chromaticity. Our theoretical results are compared favorably with tracking simulations for the long-range resistive wall instability, and we also indicate how damping and diffusion from sychrotron emission can further reduce or even stabilize the dynamics.

INTRODUCTION

Coupled-bunch instabilities, in which wakfield forces between bunches drives coupled-bunch transverse betatron oscillations, must be controlled during storage ring operation. The long-range resistive wall instability can have a large growth rate, particularly in multi-bend achromat light sources that have small aperture vacuum chamber and even smaller gap insertion device chambers. On the other hand, a high-energy storage ring such as the APS-U benefits from two important stabilizing mechanisms: first, it has reasonably large synchrotron radiation damping, and second, the relatively large chromaticity that optimizes performance provides another stabilizing force. We will discuss the physics of both of these effects in this paper, and show that they combine in the APS-U to stabilize coupled-bunch transverse instabilities by a large margin.

THEORY IN THE VLASOV LIMIT

We consider the transverse multibunch stability in the limit that the long-range dipole wakefield W_y varies linearly over the bunch length σ_z. This is sometimes true for higher-order modes, and typically valid for the long-range resistive wall instability that we will use for simulations. We define Ω to be the complex frequency of the perturbation, with $\Im(\Omega) > 0$ indicating instability, and then linearize the problem about the equilibrium distribution $\bar{f}(\mathcal{J})$, where $(\Phi, \mathcal{J})$ identify the angle and action variables in the longitudinal plane. We show in Ref. [1] that the perturbation $g_n(\Phi, \mathcal{J}; \Omega)$ of the n^{th} bunch satisfies the linearized equation

$$\begin{aligned}\mathcal{R}^{\parallel}_{\text{FP}}[g_n] = \frac{1}{ic}\left[\Omega + \frac{i}{\tau_y} + i\omega(\mathcal{J})\frac{\partial}{\partial\Phi}\right] g_n(\Phi, \mathcal{J}; \Omega) \\ - \bar{f}_n(\mathcal{J})\frac{e^{-ik_\xi z}}{ic}\sum_{j=1}^{M} \mathsf{M}_{n,j}\int d\Phi' d\mathcal{J}' e^{ik_\xi z'} g_j.\end{aligned} \tag{1}$$

Here, the right-hand side writes the longitudinal frequency $\omega(\mathcal{J})$, has used $k_\xi z = (\omega_0\xi/c\alpha_c)z$ to denote the head-tail (chromatic) phase for a ring with revolution frequency $\omega_0 = 2\pi/T_0$, chromaticity ξ, and momentum compaction α_c, τ_y for the longitudinal damping time, and introduced the wakefield matrix coupling the bunches together as

$$\mathsf{M}_{n,j} = \frac{e^2 N_{e,j}}{2\gamma mc^2 T_0}\sum_{\ell=0}^{\infty} W_y(-\ell cT_0 - L_{nj})e^{i\ell\omega_\beta T_0}, \tag{2}$$

where $N_{e,j}$ is the number of electrons in bunch j while L_{nj} is the distance between bunch n and j [2].

The left-hand side of Eq. (1) contains the longitudinal damping and diffusion due to synchrotron radiation, which we denote as the Fokker-Planck operator $\mathcal{R}^{\parallel}_{\text{FP}}$. In the usual longitudinal variables (z, p_z) this operator acts as

$$\mathcal{R}^{\parallel}_{\text{FP}}[g_n] = \frac{2}{c\tau_z}\left[\sigma_\delta^2\frac{\partial^2}{\partial p_z^2} + p_z\frac{\partial}{\partial p_z} + 1\right] g_n(z, p_z; \Omega), \tag{3}$$

where σ_δ is the equilibrium energy spread and τ_z is the longitudinal damping time. For starters we will neglect the dissipative operator $\mathcal{R}^{\parallel}_{\text{FP}}$ by setting $\tau_z \to \infty$, and focus on the effect of chromaticity on the stability.

When the right-hand side of Eq. (1) vanishes, we can invert the operator [3] in brackets to solve for g_n. Multiplying by $e^{ik_\xi z}$ and integrating over phase space results in

$$\begin{aligned}\int d\Phi d\mathcal{J} e^{ik_\xi z} g_n = \int d\Phi d\mathcal{J}\frac{\bar{f}_n e^{i\Omega\Phi/\omega(\mathcal{J})}}{\omega(e^{-2\pi i\Omega} - 1)} \\ \times\int_{\Phi}^{\Phi+2\pi} d\hat{\Phi} e^{-ik_\xi\hat{z}} e^{-i\Omega\hat{\Phi}/\omega(\mathcal{J})} \\ \times\sum_j \mathsf{M}_{n,j}\int d\Phi' d\mathcal{J}' e^{ik_\xi z'} g_j.\end{aligned} \tag{4}$$

The top two lines depend only upon equilibrium properties, while the last line involves the wakefield and the potential. This latter part can be diagonalized by going to the coupled-bunch modes τ_n, which effectively sets $\sum_j \mathsf{M}_{n,j} g_j \to \lambda_n\tau_n$, where λ_n is the coupled-bunch growth rate when $\xi = 0$. Then, the integrated perturbation can be cancelled from each side, and the result is a dispersion relation whose solution gives the complex instability frequency for a given chromaticity and $\xi = 0$ growth rate λ.

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For simplicity, here we restrict our analysis to a harmonic longitudinal potential. In this case $\omega(\mathcal{J}) = \omega_s$, $\bar{f} = e^{-\mathcal{J}/\sigma_z\sigma_\delta}/(2\pi\sigma_\delta\sigma_z)$, $z = \sqrt{2\sigma_z\mathcal{J}/\sigma_\delta}\cos\Phi$, and the dispersion relation can be written as [1,4]

$$1 = \sum_{m=-\infty}^{\infty} \frac{\lambda e^{-k_\xi^2\sigma_z^2} I_m(k_\xi^2\sigma_z^2)}{\Omega - m\omega_s}. \tag{5}$$

We plot solutions to Eq. (5) as the solid lines in Fig. 1 for four value of $k_\xi\sigma_z$ and the APS-U relevant parameters of Table 1. These compare very closely with `elegant` [5] tracking simulations, shown as the points of same color.

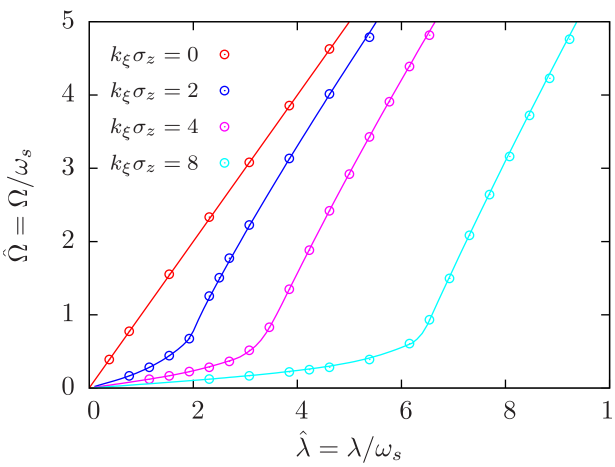

Figure 1: Effect of chromaticity on multibunch stability. The lines are the theory, while the points are results obtained from `elegant` tracking simulations. The parameters are listed in Table 1.

Interestingly, the instability shows two clear regimes: when $\Im(\Omega) \ll \omega_s$ the instability can be significantly contained by the chromaticity, and one can show that $\Im(\Omega) \approx \lambda/(\sqrt{2\pi}k_\xi\sigma_z)$. On the other hand, when $\Im(\Omega) > \omega_s$ the growth rate increases at a much faster rate. By equating these two relations, one finds that the "weak" regime pertains when the $\xi = 0$ growth rate is less than the chromatic tune shift over the bunch, $\Im(\lambda) < \xi\omega_0\sigma_\delta$. At higher wakefield strengths chromatic effects can no longer contain the instability.

INCLUDING DISSIPATIVE EFFECTS

The Fokker-Planck operator, Eq. (3), is more complicated in action-angle variables. To simplify matters somewhat we introduce the dimensionless action $r = \mathcal{J}/\sigma_z\sigma_\delta$, in terms of which the Fokker-Planck operator

$$\begin{aligned}\mathscr{R}_{\text{FP}}^{\parallel} = \frac{2}{c\tau_z}\left[r\frac{\partial^2}{\partial r^2} + (1+r)\frac{\partial}{\partial r} + \frac{1}{4r}\frac{\partial^2}{\partial\Phi^2} + 1\right] \\ - \left\{\frac{e^{2i\Phi}}{c\tau_z}\left[r\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} - \frac{1}{4r}\frac{\partial^2}{\partial\Phi^2}\right.\right. \\ \left.\left. + i\frac{\partial^2}{\partial\Phi\partial r} + i\frac{r-1}{2r}\frac{\partial}{\partial\Phi}\right] + c.c.\right\}.\end{aligned} \tag{6}$$

Table 1: Parameters Used in `elegant` Simulations

Parameter	Symbol	Value
Vertical tune	ν_y	36.1
Momentum compaction	α_c	4.04×10^{-5}
Bunch length	σ_z	16.06 mm
Energy spread	σ_δ	0.135%
Revolution time	T_0	3.68 ms
Synchrotron frequency	$\omega_s/2\pi$	160 Hz
Damping time in y	τ_y	15.4 ms
Damping time in z	τ_z	20.5 ms
Unstable eigenvalue	λ	$\Im(\lambda)(i - 0.6)$

Equation (1) with the right-hand-side given by Eq. (6) can be cast into an eigenvalue problem by expanding the perturbation as a sum of orthogonal modes as shown in Ref. [6]. Our approximate solution begins by expanding the perturbation in azimuthal (angular) modes as follows

$$\tilde{g}_n(\Phi, \mathcal{J}; \Omega) = \sum_\ell g_n^\ell(\mathcal{J}; \Omega)e^{i\ell\Phi}. \tag{7}$$

Inserting the expansion Eq. (7) into $\mathscr{R}_{\text{FP}}^{\parallel}$ results in the first line of Eq. (6) that is diagonal in the azimuthal index ℓ, while the second two lines couple the mode ℓ to those at $\ell \pm 2$. We neglect the latter coupling, which applies provided the longitudinal damping is sufficiently weak for the given chromatic phase $k_\xi\sigma_z$.

Next, we use the azimuthal expansion Eq. (7) in Eqs. (1) and (6), and isolate the modes by multiplying by $e^{-im\Phi}/2\pi$ and integrating over angle from 0 to 2π; for the simple harmonic potential we obtain

$$\begin{aligned}\left[r\frac{d^2}{dr^2} + (1+r)\frac{d}{dr} - \frac{m^2}{4r} + 1 - \frac{\tau_z(\Omega - m\omega_s)}{2i} - \frac{\tau_z}{2\tau_y}\right] g_n^m \\ = i\tau_z\frac{e^{-r}J_m(k_\xi\sigma_z\sqrt{2r})}{4\pi i^m\sigma_z\sigma_\delta}\sum_{j=0}^{M}\mathsf{M}_{n,j}\int d\Phi d\mathcal{J}e^{ik_\xi\sigma_z}g_j.\end{aligned} \tag{8}$$

We can invert the differential operator on the left-hand side of Eq. (8) using a suitable Green function. We show how to do this in Ref. [1]; after diagonalizing the wakefield matrix M we find that

$$\begin{aligned}g_n^m = \sum_{p=0}^{\infty}\frac{(k_\xi\sigma_2/\sqrt{2})^{2p+|m|}e^{k_\xi^2\sigma_z^2/2}}{\Omega - m\omega_s + i/\tau_y + i(2p+|m|)/\tau_z} \\ \times\frac{e^{-r}r^{|m/2|}L_p^{|m|}(r)}{2\pi i^m\sigma_z\sigma_\delta(p+|m|)!}\lambda\int d\Phi' d\mathcal{J}' e^{ik_\xi\sigma_z}g_n.\end{aligned} \tag{9}$$

Now, we can eliminate the perturbation g by multiplying by $e^{im\Phi}e^{ik_\xi z}$, integrating over $(\Phi, \mathcal{J})$, and summing over azimuthal index m. We find the dispersion relation [1]

$$\begin{aligned}1 = \sum_{m=-\infty}^{\infty}\sum_{p=0}^{\infty}\frac{\lambda e^{-k_\xi^2\sigma_z^2}}{p!(p+|m|)!}(k_\xi^2\sigma_z^2/2)^{2p+|m|} \\ \times\left[\Omega - m\omega_s + \frac{i}{\tau_y} + \frac{i}{\tau_z}(2p+|m|)\right]^{-1}.\end{aligned} \tag{10}$$

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The full dispersion relation, Eq. (10), appears to have an additional damping-like term $\sim (2p + |m|)/\tau_z$ in the sum. This is the same damping identified in the modal analysis of Ref. [6] that was then associated with the synchrotron mode whose radial and azimuthal numbers are (p, m). The physical origin of this mode number-dependent damping is the longitudinal diffusion, which tends to smooth out the rapidly-varying higher-order modes more effectively than those of lower order.

We illustrate the effects of the longitudinal damping and diffusion on coupled bunch stability in Fig. 2, where we plot the ratio of the Fokker-Planck growth rate $\Im(\Omega_{\text{FP}})$ derived from Eq. (10) to that of the Vlasov limit Eq. (5) as a function of the $\xi = 0$ dimensionless growth rate $\Im(\hat{\lambda})$. We see that even relatively small longitudinal damping can significantly reduce the instability growth rate in the "weak" regime where $\Im(\hat{\lambda}) < 3k_\xi\sigma_z/4$, particularly as the chromatic phase becomes large. In addition, Fig. 2 predicts that the growth rate approaches its nominal, undamped value when $\Im(\hat{\lambda}) > 3k_\xi\sigma_z/4$.

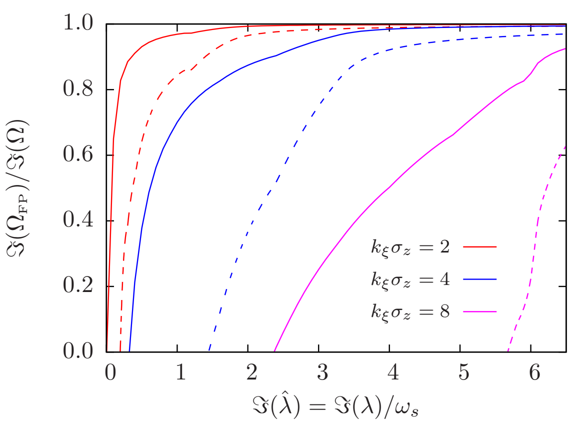

Figure 2: Ratio of the theoretical growth rate including longitudinal damping to that with $\tau_z \to \infty$ as a function of the $\xi = 0$ growth rate $\Im(\lambda)/\omega_s$. Solid lines assume $\tau_z\omega_s = 1/500$, while the dashed lines have $\tau_z\omega_s = 1/100$.

To further test our theory, we varied the longitudinal damping in the `elegant` simulations by modeling the energy loss as having two contributions: the first is merely a uniform energy loss for all particles, while the second includes the explicit damping and diffusion associated with emission. This allows us to keep τ_y constant by fixing the total energy loss, while varying the value of τ_z. Comparison of these simulations and our theory for the APS-U parameters in Table 1, $k_\xi\sigma_z = 4$, and three values of τ_z are shown in the top panel of Fig. 3. Both theory and simulation show a very similar decrease in the "weak" growth rate as the longitudinal damping rate $1/\tau_z$ increases. The difference between the two increases at larger damping rates and in the strong regime where the mode coupling in the Fokker-Planck operator becomes important.

As a final illustration, the middle and bottom panel compare the unstable perturbation when $\lambda = 2.5\omega_s$ (middle) to that with $\lambda = 3.75\omega_s$ (bottom). We find the perturbation has

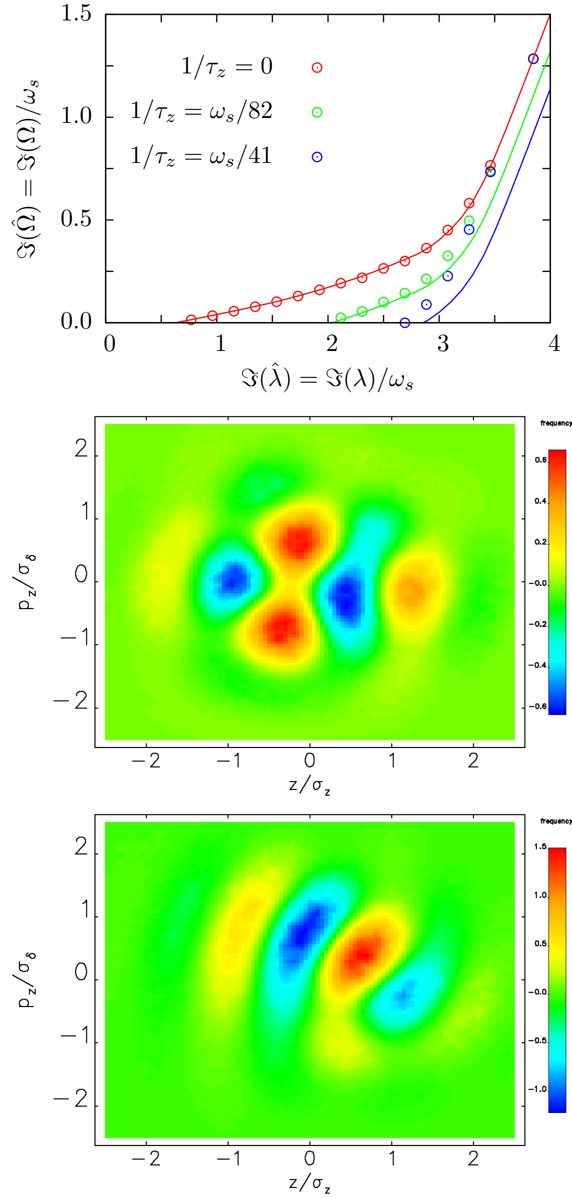

Figure 3: Effect of longitudinal damping on stability when $k_\xi\sigma_z = 4$. Right panel plots points for `elegant` simulation results, while lines are the theory. As the longitudinal damping is increased the "weak" regime of the instability is stabilized. Middle and right panel show the instability profile when $1/\tau_z = 0$ and $\lambda = 2.5\omega_s$ and $\lambda = 3.75\omega_s$, respectively.

the azimuthal symmetry of a synchrotron mode when the instability is weak, while it is clearly a mix of many modes in the strong regime.

CONCLUSIONS

We showed that chromaticity reduces transverse multi-bunch instability growth rates provided the chromatic tune spread is larger than the $\xi = 0$ growth rate. In this case synchrotron emission further damps the instability through diffusion in the longitudinal plane.

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