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NEW TECHNIQUES TO COMPUTE THE LINEAR TUNE

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Abstract

Tune determination in numerical simulations is an essential aspect of nonlinear beam dynamics studies. In particular, because it allows probing whether a given initial condition is close to resonance, and it enables assessment of the stability of the orbit, i.e. whether the motion is regular or chaotic. In this paper, results of recently developed techniques to obtain accurate tune computation from numerical simulation data are presented and discussed in detail.

INTRODUCTION

The evaluation of the tune of an orbit in a nonlinear system provides essential information on the underlying dynamics. Indeed, the properties of the tune allow classifying the orbit as regular or chaotic, and in the first case whether it is resonant or not. All these essential features can be extracted solely from the tune value, provided the approach used ensures a high-accuracy. In past years, strong efforts had been devoted to the development of algorithms for the precise tune determination, in some cases importing methods from neighbouring fields, such as celestial mechanics (see [1, 2] for a review of this approach, and [3] for a general overview). In other cases, analytical formulae had been derived, which are based on the Fourier series involved in the tune determination [4, 5].

More recently, a number of studies have been performed on the topic of quasiperiodicity of time series, and resulted in a number of novel approaches that could lead to improvements in the accuracy of the tune estimate [6–10]. These studies are not well known in the domain of accelerator physics and in this paper we present how they could be applied to beam dynamics.

Another point considered here, is the tune determination in the presence of amplitude modulation of the time series. In this case, time series cannot be represented anymore as a sum of complex exponential terms with constant coefficients. In all methods that are normally used to perform harmonic analysis of time series it is assumed that the amplitude is time independent. However, this is not always the case, as the presence of decoherence and filamentation introduce a natural amplitude modulation of the turn-by-turn signal measured. Therefore, in this paper, we will also consider this special topic, that is normally neglected, and show how the accuracy of the tune determination can be improved by means of new approaches.

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TUNE DETERMINATION FOR SIGNALS OF CONSTANT AMPLITUDE

Theory

The standard methods to determine the tune from time series with constant amplitude are based on Ref. [1] and on Refs. [4, 5], which both represent a significant improvement in terms of tune error if compared with the accuracy provided by the standard FFT-based method. The latter provides a tune error that scales as N^{-1}, with N being the length of the time series, whereas the improved methods are in the implementation with the Hanning filter, accurate as N^{-4}, which represents a huge improvement.

A further accuracy improvement can be obtained by considering the following approach based on the use of analytic filters w^n of the form

$$w^n(t) = \exp\left[-\frac{1}{t^n(1-t)^n}\right], \quad t \in]0,1[\tag{1}$$

and set to 0 for t outside $]0,1[$. We can define the Birkhoff average of a function f as

$$WB_N(f)(x) = \sum_{k=1}^{N-1} \frac{w^1(k/N)}{\sum_{k=1}^{N-1} w^1(k/N)} f(z(k)), \tag{2}$$

where $z(k)$ is a generic time series. It is immediate to generalise the definition of $WB_N(f)$ to the case in which w^n is used instead. The main result [9] is that under appropriate conditions the following holds

$$\left| WB_N(f)(x) - \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N-1} f(z(k)) \right| < c_p N^{-p} \tag{3}$$

for arbitrary value of $p \in \mathbb{N}$. This super-convergence of WB_N to the limit of the average $1/N \sum_{k=1}^{N-1} f(z(k))$ can be exploited to determine the tune. In fact, if the function f is taken so that $f(z(k)) = \theta_k$, i.e. the phase of the signal at time k, then the average represents the average phase advance, whose limit is exactly the tune. Therefore, Birkhoff averages can be used to compute the tune with a super-convergent behaviour. Note that although filters like w^n had been considered in [1], they had not used in applications.

Results of Numerical Simulations

The previous results have been applied to the tune computation of a time series. A complex signal has been built by putting together normalised values of the beam position and

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divergence, so that the time series used for the numerical tests has the following form [4, 5]

$$z(k) = e^{2\pi i \nu_0 k} + \sum_{j=1}^{4} e^{-j} e^{2\pi i \nu_0 j k}, \quad k \in \mathbb{N}, \tag{4}$$

where ν_0 represents the known value of the tune and $z = x - ip_x$. The method based on Birkhoff averages has been compared with those developed in [4, 5] that provide analytical closed-form formulae for the case of a FFT with interpolation, and the case in which a Hanning filter is applied, i.e. the terms of the time series are multiplied by a factor $2\sin^2 k/N$, to improve the tune estimate. In Fig. 1 the value of the tune error $\Delta\nu$ is shown as function of N for various reconstruction methods. The Birkhoff average with $n = 1$ (left) and $n = 2$ (right) are also shown.

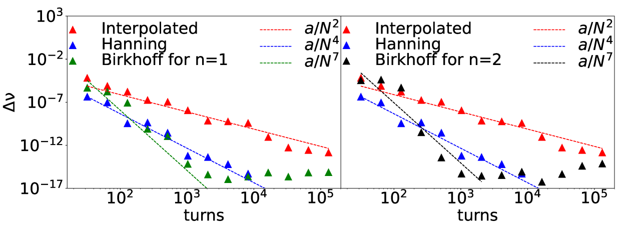

Figure 1: Comparison of the tune error $\Delta\nu$ as a function of the length of the time series N for various methods including the Birkhoff average with $n = 1$ (left) and $n = 2$ (right). The improved performance of the method based on Birkhoff average is clearly visible.

As expected, the method based on simple interpolation of the FFT and that based on the Hanning filter feature an error that scales as N^{-2} and N^{-4}, respectively. On the other hand, the error for the method based on Birkhoff averages scales as N^{-7}, with an impressive improvement of the accuracy, and saturates to the double-precision accuracy. No difference is observed between the two variants based on w^1 or w^2, as already observed in previous studies [8]. Indeed, it is related with the machine accuracy and extended precision would reveal the difference on the two variants of the Birkhoff averages.

TUNE DETERMINATION FOR SIGNALS OF VARYING AMPLITUDE

Analytical Methods

Analytical methods can be developed also for the case of signals with a time-dependent amplitude. However, the possibility to determine a closed-form expression for the tune value closely relies on the functional form of the amplitude variation. In the case of an exponential damping, i.e.

$$z(k) = e^{-\lambda k} e^{2\pi i \nu_0 k} \qquad \lambda \in \mathbb{R}^+, \tag{5}$$

it is indeed possible to achieve this goal. In this case, it can be shown that if the FFT of the time series is evaluated and its peak corresponds to the index l, then the tune can be computed as

$$\nu_0 = \frac{l}{N} + \frac{1}{\pi} \arctan\left(\frac{\xi - \mathrm{sgn}(\xi)\sqrt{\xi^2 + \tan^2\frac{\pi}{N}}}{\tan\frac{\pi}{N}} \right) \tag{6}$$

where

$$\xi = \frac{\eta + 1}{\eta - 1}, \quad \eta = \frac{\chi_+ - 1}{\chi_- - 1}, \quad \chi_\pm = \frac{|\phi(\nu_l)|^2}{|\phi(\nu_{l\pm1})|^2} \tag{7}$$

$\phi(\nu_l)$ are the FFT coefficients for $\nu_l = l/N$, and $\mathrm{sgn}(\xi) = \xi/|\xi|$ if $\xi \neq 0$. Note that a formula to compute the value of λ can also be found [11]. Similarly to the case of the time series with constant amplitude, the application of a filter can be envisaged to improve the accuracy of the tune determination. If the Hanning filter is used, then a closed-form formula can be derived [11], which can be used to determine ν_0 and λ.

Hilbert Transform

In the general case, i.e. for a time series of the form $z(k) = A(k)e^{2\pi i \nu_0 k}$ analytical methods are no longer possible. However, it is possible to use the Hilbert transform [12, 13] to evaluate the signal envelope, i.e. reconstruct $A(k)$, and then normalise the original time series to restore a situation in which the amplitude is constant over time. The Hilbert transform of a real function f is defined as [13]

$$\mathcal{H}[f](x) = \frac{1}{\pi} \mathrm{p.v.} \int_{-\infty}^{+\infty} \frac{f(y)}{x - y} \mathrm{d}y, \tag{8}$$

where p.v. stands for the Cauchy principal value. Note that the Hilbert transform can also be defined in terms of the Fourier transform of the function f [13]. The main property of the Hilbert transform that is crucial for our analysis is that the function $f(t) + i\mathcal{H}[f](t)$ is a representation of the envelope of $f(t)$. Therefore, whenever a time series features a time-dependent amplitude, it can be transformed to a new time series in which the amplitude is constant, thanks to the Hilbert transform.

Results of Numerical Simulations

The comparison in terms of error on the tune and damping factor λ for the formulae derived for a time series (5) has been performed by considering the two new formulae (with or without the Hanning filter) and the best techniques developed for the case of signals with constant amplitude. The outcome of this analysis is shown in Fig. 2 in which the tune error $\Delta\nu$ (top) and $\Delta\lambda$ (bottom) are shown as a function of the length of the time series N.

The difference in performance is clearly visible. The newly developed methods feature a $\Delta\nu$ that scales as N^{-4} or N^{-2} depending on whether the Hanning filter is used or not, respectively. This equals the optimal result that is obtained for the case of time series with constant amplitude [4, 5]. It is worth noting that when the optimal methods for constant-amplitude time series are applied to amplitude-varying time

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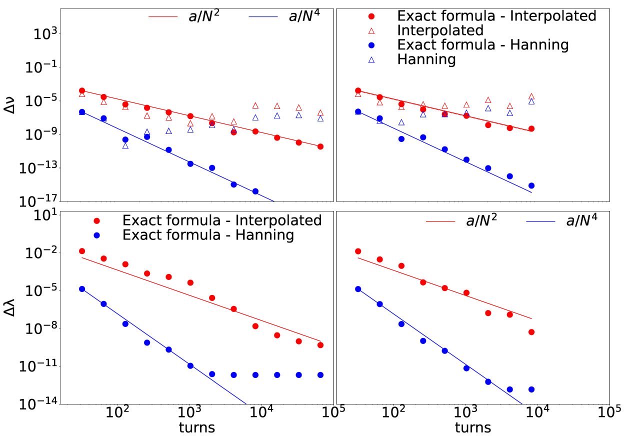

Figure 2: Comparison of $\Delta\nu$ (top) and the damping error $\Delta\lambda$ (bottom) as a function of the length of the time series N for various analytical methods. The plots differ for the value of λ, namely 5×10^{-5} (left) and 5×10^{-4}(right). The improvement brought by the new methods is clearly visible.

series, $\Delta\nu$ increases with N, which is a non-negligible drawback. The usual mitigation measure, consists in applying the methods developed for constant-amplitude time series together with a short window (of length $N_1 < N$) that is then shifted along the whole time series (with a further refinement obtained by applying the techniques developed in [14]). In this way, however, $\Delta\nu \propto N_1^{-\beta}$, which is larger than $\Delta\nu \propto N^{-\beta}$, i.e. the error of the optimised methods that can be applied to the whole time series. Note the scaling laws of $\Delta\nu$ are independent on λ.

The behaviour of $\Delta\lambda$ is shown in Fig. 2 (bottom). Obviously, only the new methods allow computing λ, and the accuracy follows the very same scaling laws of $\Delta\nu$. A saturation effect is visible, which depends on the value of λ. This can be explained by noting that any method of reconstruction is limited by the double-precision accuracy used in the numerical computations. The smaller the value of λ the smaller the value of $\bar{N}$ at which the machine precision limit is hit.

The study of the impact of the Hilbert transform has been carried out by using a time series representing the decoherence due to chromaticity that is described as [15]

$$z(k) = e^{-\alpha(k)^2/2} e^{2\pi i \nu_0 k} \qquad \alpha(k) = 2\frac{\sigma_s Q'}{\nu_s} \sin \pi \nu_s k\,, \quad (9)$$

where ν_s, σ_s, Q' are the synchrotron tune, the rms momentum spread, and the chromaticity, respectively. A typical example of this behaviour is shown in Fig. 3 (left) where the time series of Eq. (9) is shown for the case $\sigma_s Q' = 1.12 \times 10^{-4}$ and $\nu_s = 10^{-4}$. The decoherence is clearly visible (light blue), and the use of the Hilbert transform to determine the signal envelope reveals that the normalisation procedure works very well (dark blue).

The comparison of $\Delta\nu$ for the various methods is then shown in Fig. 3 (right). Globally, the direct application of the optimal methods for constant-amplitude time series

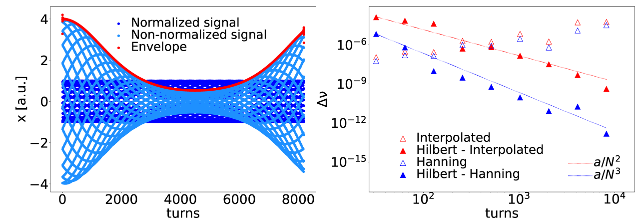

Figure 3: Left: time series including decoherence due to chromatic effects before and after normalisation with the Hilbert transform. Right: Comparison of the tune error $\Delta\nu$ as a function of the length of the time series N for various analytical methods. The improvement of the proposed method based on the Hilbert transform is clearly visible.

is hampered by the same shortcomings observed for the time series of Eq. (5), namely that $\Delta\nu$ increases with N. The application of the optimal methods [4, 5] after having normalised the original signal restores a scaling of $\Delta\nu$ as $N^{-\beta}$. It is worth noting that for the case without Hanning filter, $\beta = 2$. Therefore, the Hilbert transform fully restores the performance of the optimal method. However, when the Hanning filter is applied, $\beta = 3$. Hence, in this case some performance loss is observed, as the original performance of the optimal method, corresponds to $\beta = 4$.

CONCLUSIONS AND OUTLOOK

The precise determination of the tune of an orbit allows determining essential properties, such as whether the motion is quasiperiodic or chaotic, and in this paper a number of new techniques have been presented and discussed in detail.

The use of Birkhoff averages improves the accuracy of the tune determination for the case of time series with constant amplitude. This is a feature that would be particularly appealing to perform high-quality frequency map analysis. In fact, given the much improved tune error as a function of the length N of the time series, the number of turns required can be reduced, thus making it possible to probe many more initial conditions for the same CPU time, which would lead to a finer probing of the phase-space structure. Clearly, this increased accuracy is particularly appealing for large circular accelerators, such as the Future Circular Collider under study at CERN [16].

In the domain of time series with time-dependent amplitude, novel methods, either based on analytical closed-from formulae or on the use of the Hilbert transform, have been proposed and also in this case, the accuracy of the tune determination has been very much increased with respect to the standard techniques.

Studies of the behaviour of these new techniques in the case of frequency-modulated time series will be considered next. Moreover, the impact of noise added to the time series on the performance of these methods will be also considered. This is an essential aspect if one would like to promote the use of these approaches to the experimental domain of accelerator physics.

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