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FIRST ORDER ANALYTIC APPROACHES TO MODELLING THE VERTICAL EXCURSION FIXED FIELD ALTERNATING GRADIENT ACCELERATOR

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Abstract

Whilst the Vertical Excursion Fixed Field Alternating Gradient Accelerator (VFFA) remains a promising solution to a number of problems at the frontiers of accelerator physics [1, 2], the optics of this type of machine are still poorly understood. Current designers are forced to rely on brute-force numerical tracking codes, with optimisation dependent on time-consuming parameter scans. With an aim to both improve understanding of this machine, as well as to develop tools for rapid design and optimisation of VFFA lattices, first steps towards an analytic approach based on a linearised Hamiltonian formalism have been developed.

INTRODUCTION

Fixed Field Alternating Gradient (FFA) accelerators [3–5] in general are able to offer a number of advantages over more common types of accelerator [6]. Synchrotrons are limited in repetition rate by having to cycle magnet strength between injection and top energies. Other existing high repetition rate machines such as cyclotrons face limitations in energy reach due to relativistic effects, and in focusing strength due to lack of alternating gradient focusing. An FFA is able to overcome the limitations of both synchrotrons and cyclotrons, providing alternating gradient focusing with fixed magnetic fields – making them ideal candidates for fast acceleration, high intensity, and high repetition rate applications.

These advantages are achieved in the conventional (horizontal excursion) FFA by permitting the beam to move horizontally outwards as energy increases, replacing the time-varying fields of the synchrotron-type machines with spatially-dependent fields to achieve scaling of bending and focusing forces with energy and thereby a constant tune (in a scaling FFA). This does imply a number of concessions; a changing orbit radius necessitates changing RF frequencies as the time of revolution varies in accordance, and the fields required to realise the focusing of the FFA can in turn imply complicated coil geometries for magnet design.

By way of contrast, the vertical excursion FFA (VFFA) [1, 2, 7, 8] increases its magnetic field strength with height exponentially. Hence, stable orbits for all energies exist at the same radius, and as particles are accelerated they will adiabatically transition between these orbits (Fig. 1). This, in turn, affords several additional advantages over the horizontal FFA – including a simpler coil geometry [9]. With orbits at constant radius, the transition energy becomes infinite, and for relativistic particles the acceleration cycle becomes near-isochronous, simplifying the requirements for the RF system. This is a key advantage for muon acceleration, where high gradient cavities will be required [10] (which implies high frequency narrow-band cavities) – though removing the requirement for cycling of RF frequency in general would also enable an increase in repetition rate (and, in the case of ultrarelativistic particles, potentially even CW operation).

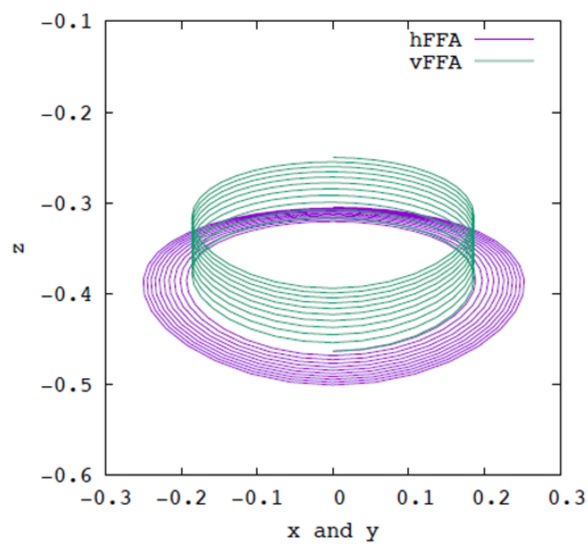

Figure 1: Diagram showing the behaviour of orbits with progressively higher and higher energies, both in the novel VFFA and conventional hFFA schemes.

However, the magnetic fields required to realise a scaling VFFA introduce additional complexity into the optics of the machine, with a high degree of nonlinearity and uniquely coupled optics. Depending on the lattice parameters, the fringe fields can contain strong longitudinal components, leading to a complex coupling of the horizontal and vertical coordinates. Moreover, the form of the closed orbits thus far identified by the numerical modelling codes employed by S. Machida, J. -B Lagrange, et al. [2], appears to be for the most part non-planar (i.e. the orbits have significant deviation from the horizontal plane over the course of a cell). These factors combine to make a machine with complicated, non-intuitive optics, and all attempts at modelling it and its properties so far have relied on extensive numerical simulation. This means that the lattice design and optimisation processes are very time-consuming, and the understanding

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of the machine and the effects of the lattice design on its ultimate properties is limited. As such, an analytic understanding of the nature of the VFFA and its fundamental behaviour as lattice parameters are varied could massively benefit the study of this type of machine.

APPROACH

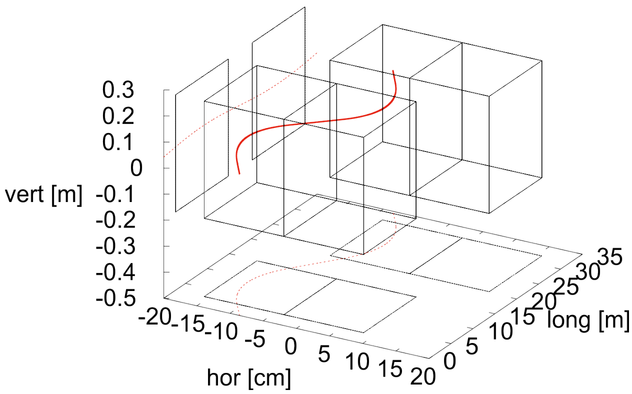

Figure 2: The geometry of an orbit through a straight VFFA cell - note that the trajectory exhibits non-zero curvature in its projection onto both the horizontal and vertical planes (dotted lines).

In order to develop an understanding of the machine, it is easiest to begin with the simplest case: a straight cell, with long magnets (in comparison to the fringe lengths). The geometry of such an orbit is shown in Fig. 2. This reduces the complexity of the system substantially: most of the behaviour of the lattice should be determined by the magnet body dynamics (assuming that the fringe fields can be neglected) and the coupling will be simplified; a small curvature within magnets should enable the curvature of the closed orbit to be neglected (if the strong focusing terms are of order $1/\rho$, the weak focusing terms from the orbit's curvature will be of order $1/\rho^2$ – which can be considered negligible for large ρ); and the change in position of the closed orbit through the magnet should be small. Moreover, this should minimise the effect of any edge angles.

With these assumptions in mind, vector potentials for the system can be derived from the scaling criterion for the VFFA,

$$B \sim e^{my}, \tag{1}$$

in which m is the vertical field index that describes the vertical scaling gradient.

By assuming a polynomial expansion and applying Maxwell's equations, we are able to obtain

$$A_Z = -\sum B_0 e^{mY} f_n \frac{X^{n+1}}{n+1} \tag{2}$$

$$A_X = 0 \tag{3}$$

$$A_Y = \sum B_0 e^{mY} \frac{1}{m} \frac{\partial f_n}{\partial Z} \frac{X^{n+1}}{n+1} \tag{4}$$

$$f_0 = f(Z) \tag{5}$$

$$f_1 = 0 \tag{6}$$

$$f_{n+2} = \frac{-1}{(n+1)(n+2)}\left(\frac{\partial^2 f_n(Z)}{\partial Z^2} + m^2 f_n(Z)\right), \tag{7}$$

where $f(Z)$ represents the field strength as a function of longitudinal position in the magnet. For the case $f(Z) = 1$ (as in the magnet body) we see that the vector potential has only a longitudinal component. To take into account the displacement of the closed orbit from the magnet midplane, x_0, it is necessary to expand this vector potential to 3rd order and make the coordinate transformation

$$X \to x - x_0(s) \tag{8}$$

$$Y \to y. \tag{9}$$

In the magnet body, then, this leads to the following Hamiltonian to first order in the dynamical variables (neglecting dipole-order and sextupole or higher order terms):

$$H = \frac{P_x^2}{2} + \frac{P_y^2}{2} \tag{10}$$

$$+ \frac{m}{\rho}\left(1 - \frac{m^2 x_0^2}{2}\right)xy \tag{11}$$

$$+ \frac{m^2 x_0}{2\rho}x^2 - \frac{m^2 x_0}{2\rho}y^2 \tag{12}$$

$$+ \frac{m^4 x_0^3}{12\rho}y^2, \tag{13}$$

in which ρ represents the curvature of the closed orbit due to the dipole field of the magnet at a given height.

Hamilton's equations are then applied to the above Hamiltonians, and transfer matrices are constructed from the resultant equations of motion via Euler integration in a simple linear optics code. This gives a quick way of testing the performance of various formulations of the Hamiltonian description of the machine in terms of tune agreement with numerical simulations, such as SCODE+.

TESTING THE LINEARISED MODEL

The behaviour of this linearised code (and thereby the Hamiltonian construction of the VFFA) is then tested by constructing a straight lattice, with parameters (15 m magnets, 35 m cell length) patterned after the 50 GeV – [energy] muon accelerator lattice proposed by S. Machida [11]. The tunes of several lattices following this scheme are evaluated, both in SCODE+ and the linearised code based on the Hamiltonians above, whilst the ratio of the field strengths at constant height between the bend and reverse bend magnets in the cell (termed the FD ratio) is varied, and the results from this are displayed in Fig. 3.

It can be seen that the tunes from the linear model and those from SCODE+ converge as the FD ratio approaches 1, and that the agreement is very good above an FD ratio of 0.95. As Fig. 4 shows, this is when the closed orbit is closest to a planar form.

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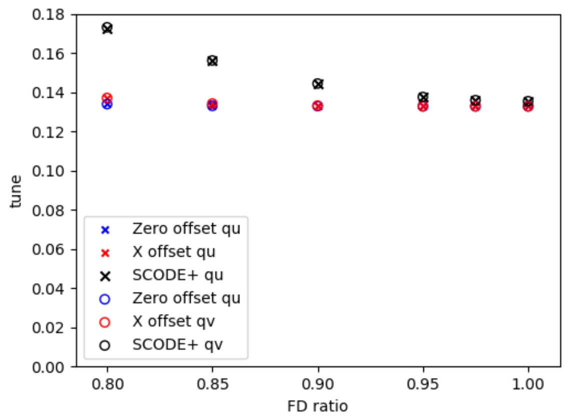

Figure 3: Tune in the decoupled u and v planes as a function of the ratio between field strengths between F and D magnets in SCODE+ (black), and the linearised Hamiltonian code with (red) and without (blue) an offset of the closed orbit from the magnet midplane considered.

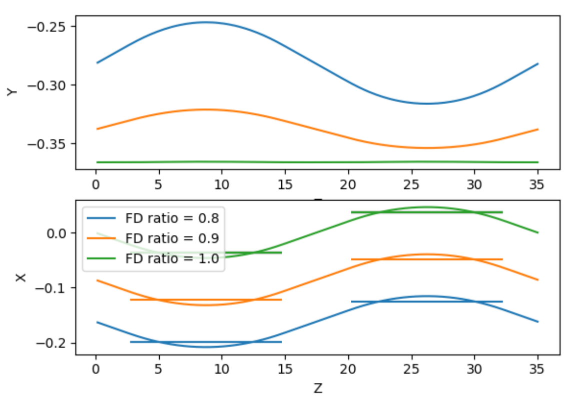

Figure 4: Orbit trajectory through the straight cell in x and y planes as a function of FD ratio. The horizontal coloured lines in the second plot represent the mean orbit x position through the magnet, used to set x_0, and the solid black horizontal line denotes the magnet midplane.

However, as the FD ratio is decreased further, the tunes from the Hamiltonian model begin to diverge from the simulation results. This divergence indicates the presence of an effect not described in the current Hamiltonian; whilst the existing Hamiltonian does contain a number of variables that should vary as a function of the FD ratio, in practice the degrees of freedom in the model as it stands seem insufficient to parameterise the system. In the magnet body Hamiltonian (Eq. (13)), there exist only two parameters that are direct properties of the lattice: ρ and x_0. Whilst at first glance it might seem that these parameters could be separately defined for the F and D magnets and thus give four degrees of freedom, in fact this is not the case. As the cell has no bending angle, ρ_f must equal $-\rho_d$ (with the difference between dipole field strengths at constant height then compensated by vertical deviation of the orbits between the two magnets, seen in Fig. 4). Similarly, to achieve zero vertical bending, the integral of the horizontal field must be zero, and as the horizontal field is directly proportional to the horizontal displacement from the magnet midplane, this necessitates $x_{0f} = x_{0d}$. We also note that successive trajectories appear as translations of each other when the FD ratio is varied (Fig. 4), showing that $\rho_f = -\rho_d$ does not vary significantly as a function of the FD ratio. The presence of x_0 terms also appears to have very little effect, despite the apparent increase in the magnitude of x_0 seen in Fig. 4. This further reinforces the conclusion that the current Hamiltonian depiction is still missing a significant element involved in the straight VFFAs as we move away from $FD = 1$.

Although fringe fields are completely neglected in the version of the lattice considered in the Hamiltonian code, it isn't immediately obvious that the simple addition of fringe field components would produce the observed tune dependence on the FD ratio. Similarly, horizontal edge angles were considered in a previous version of the code, though Fig. 4 indicates that between changes in the FD ratio the horizontal edge angle remains constant (with each successive trajectory merely translated away from the midplane). However, Fig. 4 reveals that the shape of the vertical trajectory has a dependence on the FD ratio, and therefore may have different vertical edge focusing effects for each lattice. Another potential explanation for the divergence of the tunes predicted at lower FD ratios comes from the horizontal translation of the trajectories – as the trajectory deviates further and further from the midplane, it progresses further from the most linear region of the magnet, and it may be that higher order terms in the x_0 offset must be considered.

CONCLUSION

The close numerical agreement of tune values between the linearised Hamiltonian code and the in depth SCODE+ simulations for the straight VFFA with FD ratio close to 1 is a promising first step towards an analytic formulation of the VFFA and thereby an understanding of its optics. The divergence of tune as we move away from this case toward lattices with non-equal magnet strengths (and in the future curved lattices) belies the importance of further study and the introduction of new elements into the Hamiltonian description.

One promising candidate for the missing component of the model is edge focusing from a vertical edge angle, which is one property of the orbit that displays a strong dependence on FD ratio. Hence, the next part of the analytic investigation of the VFFA will involve development of a Hamiltonian for fringe fields that accounts for this additional focusing.

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