ISBN: 978-3-95450-214-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2021-TUPAB233

DIFFUSIVE MODELS FOR NONLINEAR BEAM DYNAMICS*

C. E. Montanari†[1], A. Bazzani, Bologna University, Bologna, Italy
M. Giovannozzi, CERN Beams Department, Geneva, Switzerland
[1]also at CERN Beams Department, Geneva, Switzerland

Abstract

Diffusive models for representing the nonlinear beam dynamics in a circular accelerator ring have been developed in recent years. The novelty of the work presented here with respect to older approaches is that the functional form of the diffusion coefficient is derived from the time stability estimate of the Nekhoroshev theorem. In this paper, we discuss the latest results obtained for simple models of nonlinear betratronic motion.

INTRODUCTION

In the design stage of high-energy particle accelerators, especially those based on superconducting magnets, such as LHC [1], its upgrade project HL–LHC [2], or the proposed FCC–hh [3], it is fundamental to take into consideration the effects of unavoidable nonlinear magnetic field errors on the beam losses.

A key quantity for the evaluation of the accelerator's performance is the Dynamic Aperture (DA), i.e. the maximum amplitude of the connected phase-space region where motion is stable up to a given number of turns [4]. Studies have been carried out to characterize and predict the DA as a function of the turns particles make around the accelerator (see [5–7] and references therein), and highlighted how the evolution of DA can be described as a coexistence of weakly-chaotic regions, whose escape rates can be described in terms of a Nekhoroshev-like estimate. Such interest in DA prediction comes from the fact that DA computation requires the tracking of a large number of initial conditions for a large number of turns, typically ($N_t > 10^6$).

Recent experiments at the LHC allowed to measure the beam losses in presence of nonlinear effects, and a diffusive approach, inspired by the stability estimates of Nekhoroshev theorem for Hamiltonian systems, could allow describing in a quantitative way the long-term beam losses [8, 9]. In this paper, the theoretical background of a diffusive model based on a Fokker-Planck (FP) equation with a Nekhoroshev-like diffusion coefficient is reviewed and discussed in detail.

THE DIFFUSION FRAMEWORK

From Hamiltonian perturbation theory, we know that macroscopic diffusion phenomena have to be related to the presence of significant weakly-chaotic regions [10]. Otherwise, the presence of invariant Kolmogorov–Arnol'd–Moser tori ensure long-term stability [11].

In accelerators, the presence of a multitude of unavoidable small random perturbations [12], as well as slow modulation and transverse tune ripples, could lead to the formation of these weakly-chaotic regions. Therefore, one can assume that particle motion is described by models of the form

$$H(\theta, I, t) = H_0(I) + \xi(t) H_1(\theta, I)\,, \tag{1}$$

where (I, θ) are action-angle variables, $\xi(t)$ is a continuous stationary stochastic noise with zero mean to represent the effect of the chaotic dynamics. For a symplectic map in the neighborhood of an elliptic fixed point an optimal estimate $\|R\|$ for the Birkhoff Normal Form series reminder is given by [13, 14]

$$\|R\| = A \exp\left[-\left(\frac{I_*}{I}\right)^{\frac{1}{2\kappa}}\right]\,, \tag{2}$$

where A is a scaling factor, the exponent κ depends on the number of degrees of freedom of the system, and the action I_* represents an apparent radius of convergence of the perturbative series.

A diffusive approach for the evolution of the action distribution can be used, by application of the Averaging Principle, when the noise decorrelation is sufficiently fast (see [9] for the mathematical detail and [15] for an application to a stochastic symplectic map). The following FP equation holds for the evolution of the action distribution $\rho(I, t)$

$$\frac{\partial \rho}{\partial t} = \frac{\varepsilon^2}{2}\frac{\partial}{\partial I} D(I) \frac{\partial}{\partial I}\rho(I, t)\,, \tag{3}$$

where ε is a scaling factor related to the perturbation amplitude and, according to (2), the diffusion coefficient reads

$$D(I) = c \exp\left[-2\left(\frac{I_*}{I}\right)^{\frac{1}{2\kappa}}\right]\,, \qquad \int_0^{I_\mathrm{a}} D(I)\,\mathrm{d}I = 1\,, \tag{4}$$

where c is a normalisation constant and I_a is the position of the absorbing boundary condition, i.e. the phase-space limit beyond which a particle is considered lost or the position of a collimator. Note $D(I)$ and ρ have dimensions $I^2\,t^{-1}$ and I^{-1}, respectively.

ANALYTIC ESTIMATE OF THE CURRENT LOST

Equation (3) provides a means to obtain an analytic estimate for the current lost at the absorbing barrier (see [9] for the detail).

We start by considering the rescaled time $\tau = \varepsilon^2 t$ and by applying the following change of variables

$$x = -\int_I^{I_\mathrm{a}} \frac{1}{D^{1/2}(I')}\,\mathrm{d}I'\,, \qquad \rho'(x, \tau) = \rho(I, \tau)\frac{\mathrm{d}I}{\mathrm{d}x}\,, \tag{5}$$

* Work supported by the HL-LHC Project
† carlo.emilio.montanari@cern.ch

ISBN: 978-3-95450-214-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2021-TUPAB233

which leads to

$$\frac{\partial \rho'}{\partial \tau} = \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{1}{D^{1/2}}\frac{\mathrm{d}D^{1/2}}{\mathrm{d}x}\rho'\right] + \frac{1}{2}\frac{\partial^2 \rho'}{\partial x^2}, \tag{6}$$

where $D = D(I(x))$. By introducing the effective potential $V(x) = -\ln\left(D^{1/2}(x)\right)$, we obtain the Smoluchowsky form [16]

$$\frac{\partial \rho'}{\partial \tau} = \frac{1}{2}\frac{\partial}{\partial x}\frac{\mathrm{d}V(x)}{\mathrm{d}x}\rho' + \frac{1}{2}\frac{\partial^2 \rho'}{\partial x^2}. \tag{7}$$

Equation (7) can be made self-adjoint by means of the following change of variables

$$\rho'(x,\tau) = \exp\left[-\frac{V(x)}{2}\right]p(x,\tau), \tag{8}$$

and Eq. (7) is cast into the self-adjoint form

$$\frac{\partial p}{\partial \tau} = \frac{1}{4}\left[\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} - \frac{1}{2}\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2\right]p + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}. \tag{9}$$

The general solution of Eq. (9) can be expanded as

$$p(x,\tau) = \sum_\lambda c_\lambda(\tau)\phi_\lambda(x), \tag{10}$$

where $c_\lambda(\tau) = c_\lambda(0)e^{-\lambda\tau}$, and $\phi_\lambda(x)$ are the eigenfunctions of the operator on the r.h.s. of Eq. (9), namely

$$2\left\{-\frac{1}{4}\left[\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} - \frac{1}{2}\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2\right] - \lambda\right\}\phi_\lambda(x) = \frac{\mathrm{d}^2\phi_\lambda}{\mathrm{d}x^2}. \tag{11}$$

By using the orthogonality and completeness properties of $\phi_\lambda(x)$, the solution for an initial Dirac delta distribution $\rho'(x,0) = \delta(x - x_0)$ can be written as

$$\rho'(x,\tau) = \exp\left[\frac{V(x_0) - V(x)}{2}\right]\sum_\lambda e^{-\lambda\tau}\phi_\lambda(x_0)\phi_\lambda(x),$$

and the current lost at an absorbing boundary in $x = 0$, which in the original variables corresponds to $I = I_a$, reads

$$J(\tau) = \frac{1}{2}\frac{\partial \rho'}{\partial x}\Big|_{(0,\tau)}. \tag{12}$$

If one approximates the force with a constant drift towards the boundary condition, i.e. $V(x) \simeq -\nu x$, one can obtain an analytic solution to the eigenvalue problem in Eq. (11)

$$-2\left[\lambda - \frac{\nu^2}{2}\right]\phi_\lambda(x) = \frac{\mathrm{d}^2\phi_\lambda}{\mathrm{d}x^2}, \tag{13}$$

and if we replace this solution into Eq. (12), we obtain the expression for the current lost

$$J(x_0,\tau) = \frac{|x_0|}{\tau\sqrt{2\pi\tau}}\exp\left(-\frac{(x_0 + \frac{\nu}{2}\tau)^2}{2\tau}\right). \tag{14}$$

Note J has dimension t^{-1} and for an initial condition $\delta(x - x_0)$, the linearization of the potential near $x = x_0$ provides

$$\nu(x_0) = c^{1/2}\frac{\frac{1}{2\kappa}}{I(x_0)}\left(\frac{I_*}{I(x_0)}\right)^{\frac{1}{2\kappa}}\exp\left[-\left(\frac{I_*}{I(x_0)}\right)^{\frac{1}{2\kappa}}\right], \tag{15}$$

which can be inserted into Eq. (14), for obtaining an analytical estimate of the current lost.

Equations (14) and (15) provide inevitably an underestimate of the actual current lost, as the actual drift term is a positive-increasing function for $I_0 \ll I_*$. However, we expect a good description of the local behaviour close to the absorbing boundary condition, i.e. we obtain a good estimate of the current lost for initial distributions that are close enough to the absorbing barrier at $I = I_a$.

NUMERICAL SIMULATIONS

We analyze the diffusive process in Eq. (3) using parameters similar to those in [9]: $I_* = 21.5$, $\kappa = 0.33$, and $\varepsilon^2/2 = c$. The initial condition $\rho(I,0)$ is a uniform distribution with a sharp cut, generated by a logistic function, centered at a position $I_0 < I_a$, namely

$$\rho(I,0) = \left(1 + e^{\frac{(I-I_0)}{\ell}}\right)^{-1}, \tag{16}$$

where ℓ is chosen so that the cut is smooth enough to avoid issues in the numerical integration of Eq. (3), but sharp enough to not affect the results. In Fig. 1, we show the function $D(I)$ normalized over the interval $[0, I_a]$ as a function of I/I_* (top), and the evolution of a distribution with the position of the cut at $I_0/I_* = 0.22$ and the absorbing boundary condition at $I_a/I_* = 0.233$ (bottom). The numerical integration of (3) is performed by means of a Crank-Nicolson scheme [17] with 2500 uniform spatial samples over $[0, I_a]$, in which ℓ has been set to five times the sampling fineness.

In order to test the estimate defined in Eq. (14), applying a numerical convolution to consider that the form of the initial condition is not $\delta(x - x_0)$ but (16), and the possibility to determine I_* and κ from the simulated current, two scenarios have been considered: i) I_0 is varied in a given interval, while I_a is kept constant; ii) I_0 and I_a are varied while keeping their distance constant. In both scenarios, $I_0/I_* \in [0.18, 0.23]$, whereas in the first scenario $I_a/I_* = 0.233$, and for the second one $I_a - I_0 = 0.1$. Both scenarios are relevant for the analysis of past and the proposal of future experiments on beam-halo dynamics.

In Fig. 2, we show the performance of the reconstruction method for I_* and κ from a single simulated current profile. We observe that Eq. (14) underestimates the simulated current profile in both scenarios, but scenario ii) provides a very good reconstruction of the true current. This is not unexpected, as in this case the short distance between the logistic cut (Eq. (16)) and the absorbing boundary makes the linearization in Eq. (15) well justified. However, we also observe that the fit to reconstruct I_*, κ performs much better for scenario i), which might be related to the transient effects of the initial distribution and the numerical instabilities they generate to the fitting routine.

Considering the results of the experiments probing the beam-halo dynamics at the LHC [18], we remark that, more than the beam-loss evolution itself, the timings of the various phases of the current loss measurements (e.g. ramp-up,

ISBN: 978-3-95450-214-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2021-TUPAB233

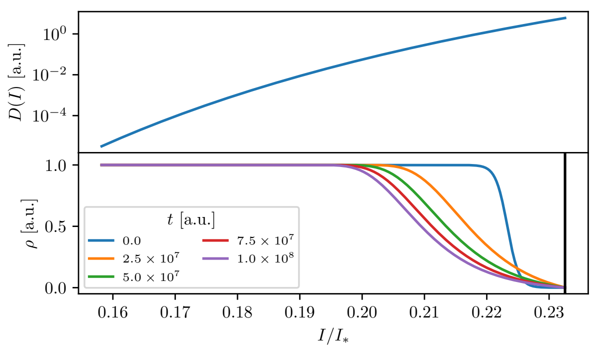

Figure 1: Top: $D(I)$ normalized, for $I_* = 21.5$, $\kappa = 0.5$. Bottom: Evolution of the distribution function. The initial condition has a logistic cut at $I_0/I_* = 0.225$. The vertical line represents the boundary condition.

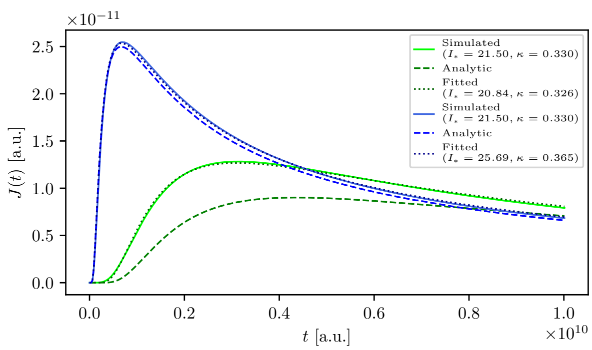

Figure 2: Comparison between simulated current, analytic formula (14) with (15) and known values of κ, I_*, and analytic reconstruction with fitted parameters. Green and blue are for scenario i) and ii), respectively. The logistic cut is at $I_0/I_* = 0.186$.

peak, decaying time) can be valid observables from the beam loss monitors to estimate the diffusion coefficient. We are interested in analyzing how accurate is the reconstruction of I_* and κ using Eq. (14), given the timing of multiple current peaks corresponding to different values of I_0/I_*.

The results of the fitting procedure are shown in Fig. 3 in terms of relative errors for κ (top) and I_* (bottom) vs I_0/I_*. The fits based on the interpolation of the entire current shape show a well-defined behaviour: scenario ii) is better in terms of the maximum error in the reconstructed parameters, although scenario i) might be superior for some specific value of I_0/I_*. At the heart of this behaviour is the linearisation of V, and remarkable is the correlation of the model parameters. However, the fit strategy based on fitting simultaneously several current-peak times recovers very accurate values of I_* and κ, and scenario ii) clearly outperforms i).

CONCLUSIONS AND OUTLOOK

A diffusive framework for the analysis of nonlinear effects on the evolution of the beam distribution has been presented and discussed using stochastic Hamiltonian systems. The key assumption is that in the weakly-chaotic regions of

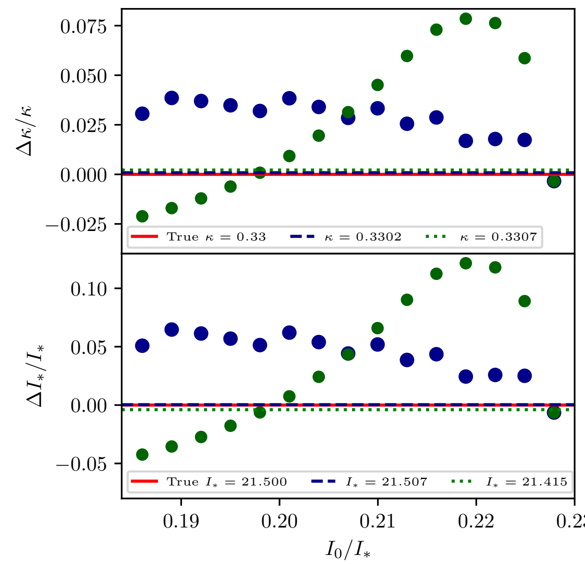

Figure 3: Relative errors for fitted parameters κ (top) and I_* (bottom) vs I_0/I_*. The circles represent the fit results based on full current shape interpolation, which are fitted individually for each I_0/I_* value. The blue dashed lines and the green dotted lines represent the fit results based on the time of the current maximum, which are considered simultaneously for all I_0/I_* values. Green and blue colors are for scenario i) and ii), respectively. The red line represents the true values of the model parameters.

phase space the action variable undergoes a diffusion process whose diffusion coefficient has a functional form derived from the stability estimates of the Nekhoroshev theorem.

Numerical simulations have been performed to assess the performance of a fitting procedure to determine the model parameters of the diffusion coefficient, which is relevant for future experiments and performance measurements on beam losses. It has been shown that the analytical estimate of the evolution of the current lost at the absorbing boundary provides a reasonable estimate of the simulated current lost. Two scenarios have been considered: the first one assumes that the position of the absorbing boundary is fixed, whereas the cut of the initial distribution is varied. The second one assumes that both the absorbing boundary and the cut vary while keeping their distance constant. The latter provides an excellent reconstruction of the model parameters when several simulations are considered together, although it was observed that the transient effects of the initial distribution might affect some fitting procedures.

Further studies are on-going to make this approach more realistic and to apply it to the collimator scans that are used to probe the beam-halo dynamics in the LHC.

REFERENCES

[1] O. Brüning (ed.) *et al.*, "LHC Design Report, Vol. I The LHC Main Ring, Ch. 17: Beam Dumping System", CERN, Geneva, Switzerland, Rep. CERN-2004-003, 2004.

ISBN: 978-3-95450-214-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2021-TUPAB233

[2] I. Béjar Alonso, O. Brüning, P. Fessia, L. Rossi, L. Tavian, and M. Zerlauth, *High-Luminosity Large Hadron Collider (HL-LHC): Technical design report*, I. Béjar Alonso, Ed. Geneva, Swtizerland: CERN, 2020.

[3] A. Abada *et al.*, "FCC-hh: The Hadron Collider: Future Circular Collider Conceptual Design Report Volume 3. Future Circular Collider", *Eur. Phys. J. Spec. Top.*, vol. 228, pp. 755–1107, 2019. doi:10.1140/epjst/e2019-900087-0

[4] E. Todesco and M. Giovannozzi, "Dynamic aperture estimates and phase-space distortions in nonlinear betatron motion", *Phys. Rev. E*, vol. 53, no. 4, p. 4067, 1996. doi:10.1103/PhysRevE.53.4067

[5] M. Giovannozzi, W. Scandale, and E. Todesco, "Inverse logarithm decay of long-term dynamic aperture in hadron colliders", in *Proc. 17th Particle Accelerator Conf. (PAC'97)*, Vancouver, Canada, May 1997, pp. 1445–1447. doi:10.1109/PAC.1997.750721

[6] M. Giovannozzi, "A proposed scaling law for intensity evolution in hadron storage rings based on dynamic aperture variation with time", *Phys. Rev. ST Accel. Beams*, vol. 15, p. 024 001, 2012. doi:10.1103/PhysRevSTAB.15.024001

[7] A. Bazzani, M. Giovannozzi, E. H. Maclean, C. E. Montanari, F. F. Van der Veken, and W. Van Goethem, "Advances on the modeling of the time evolution of dynamic aperture of hadron circular accelerators", *Phys. Rev. Accel. Beams*, vol. 22, p. 104 003, Oct. 2019. doi:10.1103/PhysRevAccelBeams.22.104003

[8] A. Bazzani, O. Mazzarisi, M. Giovannozzi, and E. Maclean, "Diffusion in stochastically perturbed hamiltonian systems with applications to the recent lhc dynamic aperture experiments", in *Nonlinear Dynamics and Collective Effects in Particle Beam Physics*, M. C. Swapan Chattopadhyay and S. D. Mitri, Eds. Hackensack, NJ, USA: World Scientific, 2019, pp. 70–85.

[9] A. Bazzani, M. Giovannozzi, and E. Maclean, "Analysis of the non-linear beam dynamics at top energy for the CERN Large Hadron Collider by means of a diffusion model", *Eur. Phys. J. Plus*, vol. 135, no. 1, p. 77, 2020. doi:10.1140/epjp/s13360-020-00123-2

[10] C. Froeschlé and E. Lega, "Weak chaos and diffusion in hamiltonian systems", in *The Dynamics of Small Bodies in the Solar System*, B. A. Steves and A. Roy, Eds. Berlin, Heidelberg, Germany: Springer, 1999, pp. 463–502.

[11] A. Morbidelli and A. Giorgilli, "On a connection between KAM and Nekhoroshev's theorems", *Physica D*, vol. 86, no. 3, pp. 514–516, 1995. doi:10.1016/0167-2789(95)00199-E

[12] J. A. Ellison, "Noise effects in accelerators", in *Beam Measurement*. Hackensack, NJ, USA: World Scientific, 1999, pp. 428–450.

[13] G. Turchetti, "Nekhoroshev stability estimates for symplectic maps and physical applications", in *Number theory and physics*, J. M. Luck, P. Moussa, and M. Waldschmidt, Eds. Berlin, Heidelberg, Germany: Springer, 1990, pp. 223–234.

[14] A. Bazzani, S. Marmi, and G. Turchetti, "Nekhoroshev estimate for isochronous non resonant symplectic maps", *Cel. Mech.*, vol. 47, p. 333, 1990. doi:10.1007/BF00051010

[15] C. E. Montanari, "Diffusive approach for non-linear beam dynamics in a circular accelerator", M.S. thesis, Department of Physics and Astronomy, Alma Mater Studiorum, University of Bologna, 2019.

[16] R. Hannes and F. Till, *The Fokker-Planck Equation, Methods of Solution and Applications*. Berlin, Heidelberg, Germany: Springer, 1996.

[17] J. Crank and P. Nicolson, "A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type", in *Math. Proc. Camb. Philos. Soc.*, Cambridge, UK, Cambridge University Press (CUP), Jan. 1947, pp. 50–67. doi:10.1017/s0305004100023197

[18] A. Gorzawski *et al.*, "Probing LHC halo dynamics using collimator loss rates at 6.5 TeV", *Phys. Rev. Accel. Beams*, vol. 23, no. 4, p. 044 802, 2020. doi:10.1103/PhysRevAccelBeams.23.044802