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ANALYSIS OF MULTIBUNCH SPECTRUM FOR AN UNEVEN BUNCH DISTRIBUTION IN A STORAGE RING*

R. Li, F. Marhauser, Jefferson Lab, Newport News, Virginia, USA

Abstract

Modern storage-ring designs often require an uneven bunch distribution pattern. An uneven bunch fill pattern can result in complex structures for the beam current spectra. Particularly at high average beam currents, these complex current spectra need to be taken into account in concern of beam-dynamical effects. In this study, we analyze a beam current spectrum for various filling patterns with bunch trains and gaps. The characteristics of the resulting beam current spectra are illustrated and discussed.

INTRODUCTION

The performance goals of modern storage rings, either high brilliance for synchrotron light sources or high luminosity for circular colliders, often require a high average-current operation using a large number of bunches. Such a parameter regime is prone to a fast growth of coupled-bunch instabilities, which demands efficient mitigation measures. The RF buckets in these storage rings are often unevenly filled, because of a gap or several gaps needed for the injection/ejection and the requirement of ion clearing in the electron ring or the electron-cloud clearing in the ion ring. The impact of a partial-fill pattern on the potential well distortion and coupled-bunch instabilities has been a topic of interest for a long time [1-5]. An analytical expression of the beam current spectrum for an uneven bunch fill pattern should be helpful for these studies. Such an expression was presented earlier for a general bunch filling pattern [6].

Recently, during the Jefferson Lab electron-ion collider (JLEIC) design study, numerical modeling of the beam power loss was carried out for the JLEIC e-ring, specifically due to the IR chamber impedance [7], and for the circulator cooler ring of the JLEIC e-cooler due to the harmonic kicker impedance [8]. To understand the intricate results from these studies, we analyzed the beam current spectrum for the simple situation when the beam consists of bunch trains (for constant bunch charge) separated by gaps. Our analysis is described in this paper. In particular, using the JLEIC examples, we show the roles of the filling factor and the sideband interference and demonstrate how their combination gives rise to the intricate behavior of the current spectrum. Good agreement between theory and the direct discrete Fourier transform (DFT) is exhibited. The current spectral amplitude for the ongoing electron-ion collider (EIC) design is also presented.

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BUNCH DISTRIBUTION

We first describe the uneven bunch distribution in a storage ring and its normalization. Time duration, rather than distance, is used here for the description.

Let f_0 be the revolution frequency, with revolution period $T_0 = f_0^{-1}$. The ring is filled by K bunch trains (or pulses), and the time duration for each bunch train is T_{tr}, so $T_0 = KT_{tr}$. Each bunch train consists of M_0 evenly-spaced bunch slots with a bunch spacing T_B, or $T_{tr} = M_0 T_B$. In the even-fill scenario, the total number of bunches in the ring is $M = KM_0$, and the bunch repetition rate is $f_{rep} = T_B^{-1}$ with $\omega_B = 2\pi f_{rep}$. For this study, we are interested in the case when only the first N_0 slots in each bunch train are filled with bunches of equal bunch charge, followed by $N_g = (M_0 - N_0)$ slots of missing bunches, as illustrated in Fig. 1. The time duration in each bunch train filled by the bunches is $T_{fill} = N_0 T_B$. and the filling factor is

$$\alpha = T_{fill}/T_{tr} = N_0/M_0,$$

the repetition rate for bunch trains is $f_{tr} = T_{tr}^{-1}$ with $\omega_{tr} = 2\pi f_{tr}$.

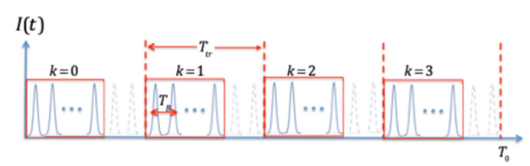

Figure 1: Bunch distribution pattern for four bunch trains ($K = 4$) in a ring.

Let the normalized density distribution for each bunch be $\lambda(t)$. For the rms bunch length σ_t much shorter than the bunch spacing, $\sigma_t \ll T_B$, we have

$$\int_{-T_B/2}^{T_B/2} \lambda(t)dt \approx \int_{-\infty}^{\infty} \lambda(t)dt = 1.$$

For an uneven-fill beam distribution pattern, as depicted in Fig. 1, the density distribution function $F(t)$ is a periodic function

$$F(t) = \sum_{k=0}^{K-1} \sum_{n=0}^{N_0-1} \lambda(t - T_B/2 - nT_B - kT_{tr}),$$

with periodic condition $F(t) = F(t + T_0)$ and normalization $\int_0^{T_0} F(t)dt = \alpha M$. For N_b number of charged particles (with unit charge e) in each bunch, the total beam current is $I(t) = e\, N_b\, F(t)$, and the average beam current is

$$I_{ave} = e\alpha\, N_b/T_B.$$

We can view $F(t)$ as the product of an evenly-distributed periodic bunch distribution $E(t)$ and an evenly-distributed periodic rectangular function $R(t)$

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$$F(t) = E(t) \cdot R(t) \tag{1}$$

with

$$E(t) = \sum_{m=0}^{M_0-1} \lambda(t - T_B/2 - m\,T_B), \tag{2}$$
$$R(t) = \sum_{k=0}^{K_0-1} R_0\left(t - T_{fill}/2 - m\,T_{tr}\right), \tag{3}$$

and

$$E(t) = E(t + T_0), \qquad R(t) = R(t + T_0).$$

Here $R_0(t)$ is the basic rectangular function describing a bunch train

$$R_0(t) = \begin{cases} 1 & (|t| < T_{fill}/2) \\ 0 & (T_{fill}/2 \le |t| \le T_{tr}/2)' \end{cases}$$

as illustrated in Fig. 2. For a single bunch train in the ring, $K = 1$, Eq. (1) is illustrated in Fig. 3.

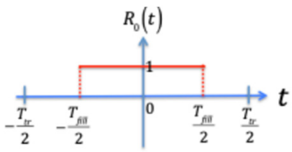

Figure 2: Basic rectangle function.

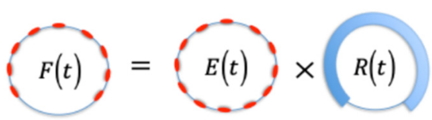

Figure 3: Illustration of a bunch train with one gap in a ring.

Define the Fourier transform of a function $g(t)$ as

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{g}(\omega) e^{-i\omega t} d\omega, \quad \tilde{g}(\omega) = \int_{-\infty}^{\infty} g(t) e^{i\omega t} dt,$$

and let the Fourier spectra for $E(t)$ and $R(t)$ be $\tilde{E}(\omega)$ and $\tilde{R}(\omega)$ respectively. The Fourier spectrum for their product $F(t) = E(t)R(t)$ is then

$$\tilde{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{E}(\omega')\tilde{R}(\omega - \omega') d\omega' . \tag{4}$$

FOURIER SPECTRUM

Next, we give the expression of $\tilde{F}(\omega)$ in Eq. (4) by first analyzing $\tilde{E}(\omega)$ and $\tilde{R}(\omega)$ respectively.

For an even fill of M bunches in the ring, the distribution function $E(t)$ in Eq. (2) can be written as a convolution of the single bunch distribution function $\lambda(t)$ and a periodic function $s_E(t)$ of δ-summation

$$E(t) = \int_0^{T_0} dt' \lambda(t - t') s_E(t' - t_E), \tag{5}$$

with time shift $t_E = T_B/2$, and

$$s_E(t) = \sum_{m=0}^{M-1} \delta(t - mT_B), \quad s_E(t) = s_E(t + T_0).$$

By writing $s_E(t)$ in the form of Dirac-comb function in the time domain

$$s_E(t) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} \delta(t - mT_B - n\,T_0),$$

one gets its Fourier spectrum as a Dirac-comb function in the frequency domain

$$\tilde{s}_E(\omega) = \omega_B \sum_{p=-\infty}^{\infty} \delta(\omega - p\omega_B).$$

We then have

$$\tilde{E}(\omega) = \tilde{\lambda}(\omega)\omega_B \sum_{p=-\infty}^{\infty} (-)^p \delta(\omega - p\omega_B). \tag{6}$$

For a Gaussian distribution $\lambda(t) = e^{-t^2/2\sigma_t^2}$, its Fourier spectrum is $\tilde{\lambda}(\omega) = e^{-\omega^2\sigma_t^2/2}$.

Similar to $E(t)$, the periodic rectangular bunch-train function $R(t)$ in Eq. (3) can be written as

$$R(t) = \int_0^{T_0} dt' R_0(t - t') s_R(t' - t_R) \tag{7}$$

with $t_R = T_{fill}/2$, and

$$s_R(t) = \sum_{k=0}^{K-1} \delta(t - kT_{tr}), \quad s_R(t) = s_R(t + T_0).$$

This periodic δ-summation can be written as a Dirac-comb in the time domain

$$s_R(t) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{K-1} \delta(t - kT_{tr} - n\,T_0),$$

and its Fourier transform is

$$\tilde{s}_R(\omega) = \omega_{tr} \sum_{p=-\infty}^{\infty} \delta(\omega - l\omega_{tr}).$$

The Fourier spectrum of $R(t)$ is then

$$\tilde{R}(\omega) = \tilde{R}_0(\omega)\omega_{tr} \sum_{l=-\infty}^{\infty} e^{-i\alpha l\pi} \delta(\omega - l\omega_{tr}) \tag{8}$$

with

$$\tilde{R}_0(\omega) = \frac{2}{\omega}\sin\left(\frac{\omega T_{fill}}{2}\right).$$

The final beam current spectrum for $I(t) = e\,N_b\,F(t)$ is obtained from Eqs. (4), (6), and (8),

$$\tilde{I}(\omega) = \sum_{p=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} J(p,l)\delta(\omega - p\omega_B - l\omega_{tr}),$$

for

$$J(p,l) = (-)^p \lambda(p\omega_B)\,\mathrm{sinc}(\alpha l\pi)\,e^{-i\alpha l\pi} \tag{9}$$

with

$$\mathrm{sinc}(x) = \sin(x)/x.$$

For $f = \omega/2\pi$ as an independent variable, the normalized current spectral amplitude is

$$G(f) = \left|\tilde{I}(f)\right|/I_{ave}.$$

Note there are $l_{max} = M/K$ number of sidebands in between the adjacent harmonics of the bunch repetition rate. Let's look at the l-th side band next to the p-th harmonic. For each l between $l = 0$ and $l_{max} - 1$, the current spectrum is composed of the sidebands from all the harmonics away from the p-th harmonic, i.e., the (l-ql_{max})-th sideband of the (p+q)-th harmonic for the q-th harmonic away from the p-th harmonic for all q ($-\infty < q < \infty$). This can be seen from

$$G(f) = \sum_{p=-\infty}^{\infty} \sum_{l=0}^{l_{max}-1} H(p,l)\delta\left(f - pf_{rep} - lf_{tr}\right), \tag{10}$$

with

$$H(p,l) = \left|\sum_{q=-\infty}^{\infty} J(p+1, l - ql_{max})\right|.$$

Let us denote the weight function for the amplitude of the l-th sideband of the p-th harmonic in Eq. (9) be $w(l)$:

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$$w(l) = \mathrm{sinc}(\alpha l \pi),$$

then the normalized spectral amplitude for all the sidebands of the p-th harmonic is

$$G_0(p,f) = \sum_{l=-\infty}^{\infty} |J(p,l)|\, \delta(f - pf_{rep} - lf_{tr}), \qquad (11)$$

for $|J(p,l)| = \lambda(2\pi p f_{rep})\, w(l)$. Examining the behavior of $w(l)$ and $G_0(p,f)$ can help us better understand the final current spectrum $G(f)$ in Eq. (10) for the uneven bunch fill.

For an even-fill pattern, we have $\alpha = 1$ and $\mathrm{sinc}(\alpha l \pi) = \delta(l)$. Hence Eq. (10) is reduced to the familiar result

$$G(f) = \sum_{p=-\infty}^{\infty} \tilde{\lambda}(2\pi p f_{rep})\, \delta(f - pf_{rep}). \qquad (12)$$

BEHAVIOR OF SPECTRUM AMPLITUDE

To examine the behavior of the normalized current spectral amplitude and its dependence on the filling pattern, we consider the case of a bunch distribution pattern in the JLEIC electron ring [9], with circumference C = 2366 m and $f_{rep} = 476.3$ MHz. First, for an even bunch fill of Gaussian bunches, $M_0 = 3759$ and $\sigma_z = 9$ mm, Fig. 4 shows how $\tilde{\lambda}(\omega)$is sampled at harmonics of bunch repetition rate.

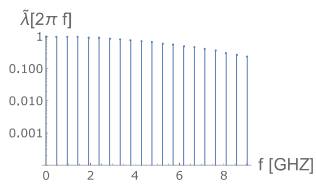

Figure 4: Normalized current spectrum for an even-fill.

We then computed the cases of a single bunch train (K = 1) with gap sizes $N_g = 1$ and $N_g = 267$, the latter being the JLEIC design value. Figure 5 shows how each harmonic of the bunch repetition rate in Fig. 4 is surrounded by a plethora of sidebands at harmonics of the bunch-train repetition rate, with the weight function $w(l)$ for each sideband. Here the weight function $w(l)$ is plotted as a continuous function of l (green curves) and as a discrete function of l (red dots), respectively. It is clear that, as the α value moves further away from unity, the plethora of sidebands around each harmonic exhibits a more intricate behavior.

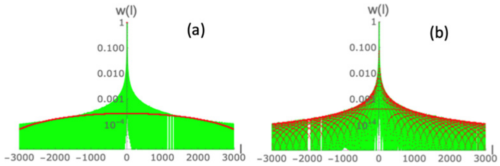

Figure 5: $w(l)$ for (a) $N_g = 1$ and (b) $N_g = 267$.

Next, the function $G_0(p,f)$ in Eq. (11) is plotted in Fig. 6 for a range of harmonic numbers p. These functions show that as the gap size increases, or the filling factor decreases, the sideband structure adjacent to each harmonic of the bunch repetition rate becomes denser and more complicated. Notice how the sidebands for each harmonic of the bunch repetition rate penetrate to adjacent harmonics.

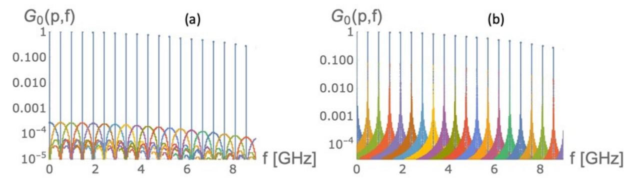

Figure 6: $G_0(p,f)$ for (a) $N_g = 1$ and (b) $N_g = 267$.

The final normalized current spectral amplitude in Eq. (10) is plotted in Fig. 7 for the two cases of interest, which accounts for the interference of sidebands from different harmonics. These analytical results are in good agreement with their counterparts in Fig. 8 obtained from numerical DFT calculations.

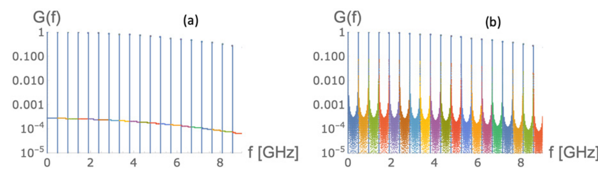

Figure 7: $G(f)$ for (a) $N_g = 1$ and (b) $N_g = 267$.

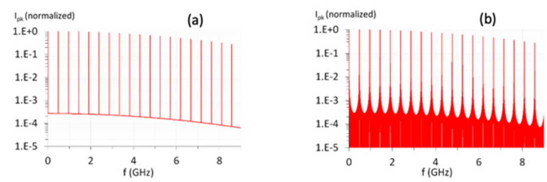

Figure 8: Numerical results of discrete Fourier transforms for the analytical cases shown in Fig. 7.

Our last example is the current spectrum for the EIC design [10], with a filling pattern of a bunch train of 1160 Gaussian bunches ($\sigma_z = 7$ mm) followed by a gap of 100 bunch spacing. Good agreement of our analytical result of the normalized current spectral amplitude for EIC and its direct DFT result is given in Fig. 9.

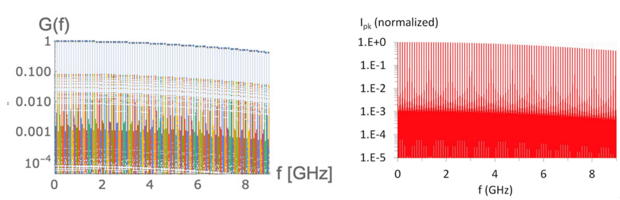

Figure 9: Current spectral amplitude for the EIC design: (a) analytical results, (b) results from direct DFT.

CONCLUSION

In this study, we analyze the beam-current spectrum for uneven fill patterns and illustrate their behavior using the JLEIC and EIC parameters as examples. The dependence of the current-spectra amplitude on the gap length is shown, and the intricate features of sideband s and their interference are discussed. Comparisons between the analytical results and the direct numerical discrete Fourier transform yields good agreement. Our analytical result agrees well with the results in Ref. [6] for the special filling pattern of bunch trains with constant bunch charge followed by gaps. Future applications of these results in the study of coupled-bunch instabilities are envisioned.

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