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MATCHING OF INTENSE BEAM IN SIX-DIMENSIONAL PHASE SPACE*

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Abstract

Beam matching is a common technique that is routinely employed in accelerator design to minimize beam losses. Despite being widely used, a full theoretical understanding of beam matching in 6D phase space remains elusive. Here, we present an analytical treatment of 6D beam matching of a high-intensity beam onto an RF structure. We begin our analysis within the framework of a linear model, and apply the averaging method to attain a matched solution for a set of bunched beam envelope equations. We then consider the nonlinear regime, where the beam size is comparable with the separatrix size. Starting with a Hamiltonian analysis in 6D phase space, we attain a self-consistent beam profile and show that it is significantly different from the commonly used ellipsoidal shape. Subsequently, we analyze the special case of equilibrium with equal space charge depression between all degrees of freedom. A comparison of beam dynamics for equipartitioned, equal space charge depression, and equal emittances beams is given.

BEAM ENVELOPES

Beam matching is attributed to finding a periodic solution for beam envelopes in periodic accelerator structures. Within the framework of the averaging method, transverse envelopes of matched beam in a quadrupole focusing accelerator structure can be written as

$$R_x(z) = R[1+\upsilon_{\max}\sin(2\pi\frac{z}{S})], \quad (1)$$

$$R_y(z) = R[1-\upsilon_{\max}\sin(2\pi\frac{z}{S})], \quad (2)$$

where R is the averaged transverse beam envelope, S is the focusing period and $\upsilon_{\max}$ is the ripple factor defined by specifics of focusing structure. Averaged transverse and longitudinal beam envelopes are determined from the set of matched envelope equations for uniformly charged ellipsoid:

$$R\frac{\mu_s^2}{S^2} - \frac{\varepsilon^2}{(\beta\gamma)^2 R^3} - \frac{3}{2}\frac{I}{I_c(\beta\gamma)^3 R}(\frac{\beta\lambda}{R_z})(1-M_z) = 0\,, \quad (3)$$

$$R_z\frac{\mu_{oz}{}^2}{S^2} - \frac{\varepsilon_z^2}{(\beta\gamma^3)^2 R_z^3} - 3\frac{I}{I_c}\frac{\beta\lambda}{(\beta\gamma)^3 R^2}M_z = 0\,, \quad (4)$$

where ε, ε_z are the 5-rms transverse and longitudinal normalized beam emittances, β and γ are beam velocity and energy, I is the beam current, $I_c = 4\pi\varepsilon_o mc^3/q$ is the characteristic beam current, m and q are the mass and charge of particles, λ is the RF wavelength, M_z is the ellipsoid coefficient, $\mu_s = \mu_o\sqrt{1-\mu_{oz}^2/(2\mu_o^2)}$ is the phase advance of transverse oscillations of a synchronous particle at the period of focusing structure S in presence of an RF field, μ_o and μ_{oz} are the phase advances of transverse and longitudinal oscillations per focusing period, respectively. The solution for the transverse equilibrium beam size, R, depends on longitudinal equilibrium beam size, R_z, and the solution for R_z in turn depends on R. Equations (3) and (4) can be re-written as $\varepsilon = \beta\gamma\,\mu_t R^2/S$, $\varepsilon_z = \beta\gamma^3\mu_z R_z^2/S$, where depressed transverse, μ_t, and depressed longitudinal, μ_z, phase advances per focusing period are:

$$\mu_t^2 = \mu_s^2 - \frac{\rho}{\rho_c}(\frac{1-M_z}{2}),\quad \mu_z^2 = \mu_{oz}^2 - \frac{\rho}{\rho_c}M_z\,, \quad (5)$$

$\rho = 3I\lambda/(4\pi c R^2 R_z)$ is the space charge density of the ellipsoidal bunch, and $\rho_c = I_c\beta^2\gamma^3/(4\pi c S^2)$ is the characteristic space charge density.

6D BEAM MATCHING

The equilibrium beam sizes are determined from simultaneous solution of Eqs. (3) and (4), which can be presented as

$$(\frac{R}{R_{ot}})^4 - 2b_t(\frac{R}{R_{ot}})^2 - 1 = 0,\quad (\frac{R_z}{R_{oz}})^4 - b_z(\frac{R_z}{R_{oz}})^3 - 1 = 0\,, \quad (6)$$

where we have introduced equilibrium beam sizes with vanishing current $I = 0$

$$R_{ot} = \sqrt{\frac{\varepsilon S}{\beta\gamma\,\mu_s}},\quad R_{oz} = \sqrt{\frac{\varepsilon_z S}{\beta\gamma^3\mu_{oz}}}\,, \quad (7)$$

and transverse, b_t, and longitudinal, b_z, space charge parameters

$$b_t = \frac{3}{2}\frac{I}{I_c\beta\gamma}(\frac{R_{ot}}{\varepsilon})^2(\frac{\beta\lambda}{2R_z})(1-M_z), \quad (8)$$

$$b_z = 3\gamma^3 M_z\frac{I}{I_c}\frac{\lambda R_{oz}^3}{R^2\varepsilon_z^2}. \quad (9)$$

Equation (6) for transverse beam size is satisfied by $R = R_{ot}\sqrt{b_t + \sqrt{1+b_t^2}}$. Together, Eqs. (6), and (7), and (8), and (9) determine matched beam sizes R, R_z through given normalized beam emittances, ε, ε_z, beam current I

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, beam momentum $\beta\gamma$ in a linac with wavelength λ and undepressed phase advances μ_o, μ_{oz} per focusing period.

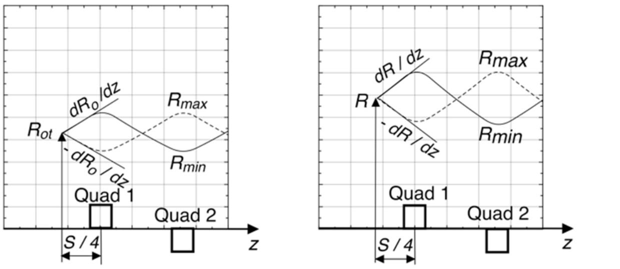

Figure 1: Transverse matching of the beam: (left) with negligible current, (right) with high-current.

Upon finding equilibrium beam sizes, R, R_z, Eqs. (1) and (2) provide matched conditions for oscillating beam envelopes. Let us note, that the solutions to Eq. (6) exist for any combination of beam and structure parameters as long as depressed phase advances, Eq. (5), are $\mu_t \geq 0$, $\mu_z \geq 0$.

Figure 1 illustrates matching of the beam with the accelerator when the beam is at the distance of $S/4$ from the middle of the first quadrupole. At this position, the beam has equal transverse sizes $R_x = R_y = R$, with slopes of beam envelopes defined as $dR_x/dz = 2\pi\upsilon_{\max}(R/S)$, $dR_y/dz = -2\pi\upsilon_{\max}(R/S)$. Deviation of beam envelopes from matched conditions results in un-periodic beam oscillations which manifests in an additional growth of phase space occupied by the beam.

Among the infinitely large number of possible matched beam solutions, the system of Eq. (6) has a specific solution for equilibrium bunch shape [1]:

$$M_z = \frac{\mu_{oz}^2}{2\mu_o^2}, \tag{10}$$

when space charge depression of particle oscillations is the same in transverse and longitudinal directions:

$$\frac{\mu_t^2}{\mu_s^2} = \frac{\mu_z^2}{\mu_{oz}^2} = 1 - \frac{3}{2\mu_o^2}\frac{I}{I_c(\beta\gamma)^3}\left(\frac{\beta\lambda}{R_z}\right)\left(\frac{S}{R}\right)^2. \tag{11}$$

The relationship between ratio of beam emittances and that of beam sizes for equal space charge depression beam is:

$$\frac{\varepsilon}{\varepsilon_z} = \left(\frac{R}{\gamma R_z}\right)^2\sqrt{\frac{1-M_z}{2M_z}} \approx \sqrt{\frac{3}{2}}\left(\frac{R}{\gamma R_z}\right)^{3/2}\sqrt{1-\frac{R}{3\gamma R_z}}, \tag{12}$$

where we used the approximation for an ellipsoidal coefficient $M_z \approx R/(3\gamma R_z)$. Figure 2 illustrates the correlation between beam radii and beam emittances for beams with equal space charge depression, Eq. (12), and that pertaining to the well-known equipartitioning condition $\varepsilon/\varepsilon_z = R/(\gamma R_z)$ [2]. As can be readily seen in the Fig. 2, the equal space charge depression condition, Eq. (12), is different from the equipartitioning condition, although two are close to one-another.

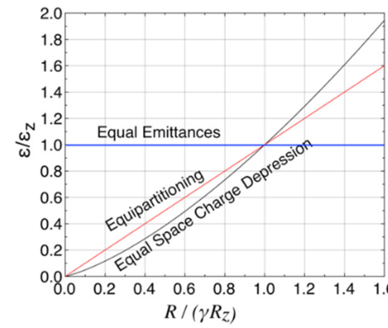

Figure 2: Ratio of beam emittances versus ratio of beam sizes for equipartitioning, equal space charge depression, and equal emittances modes.

SIMULATION OF MATCHED BEAM

An important advantage of the developed analytical framework is that it allows for a meaningful comparison between various beam equilibria, and their effects on beam phase-space growth. Figure 3 illustrates results of beam dynamic study an accelerator with constant value of longitudinal space charge depressed factor $\mu_z/\mu_{oz} = 0.2$ and variable transverse space charge depressed factor μ_t/μ_s for equipartitioned, equal space charge depression, and equal emittances beams. Figure 3a illustrates relative growth of the 6D phase space volume, $V_6 = \sqrt{\det\sigma}$, where σ is the six-dimensional sigma matrix of the beam. As can be seen, 6D phase space volume growth is correlated with the variation of transverse space charge depression (see Fig. 3b).

Phase space growth was approximately constant in case of equal space-charge depression, but varied along equipartitioning and equal emittances conditions following variation of ratio μ_t/μ_s. The strong space depression ~ 0.2, selected for simulations, corresponds to area where resonant islands in stability charts disappear, and emittance growth due to "free energy" effect is dominant [3]. Simulations indicate that within the chosen range of parameters, minimization of six-dimensional phase space volume growth is not related to any specific beam equilibrium, like equipartitioning, but is instead related to minimization of space charge depression.

(a) (b)

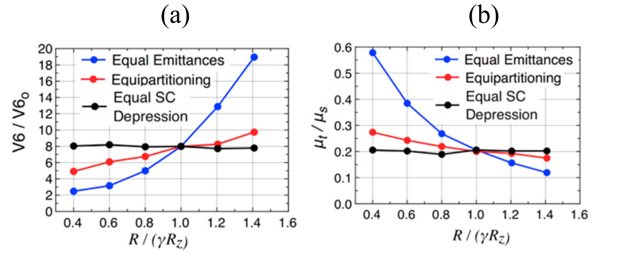

Figure 3: (a) growth of six-dimensional phase space, (b) transverse space charge depression factor for various beam equilibria.

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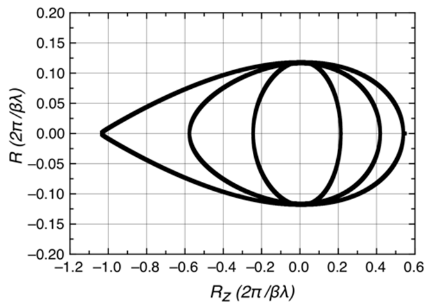

Figure 4: Matched self-consistent beam profile for various longitudinal beam sizes,

SELF CONSISTENT BEAM DYNAMICS

Analysis of beam matching presented above was performed within a linear approximation of particle dynamics around a synchronous trajectory with the additional approximation of beam space charge field corresponding of that of a uniformly charged ellipsoid. In Ref. [1], the self-consistent analysis of high-intensity bunched beam equilibrium was performed for the case where the beam size is limited by the separatrix size. In such a case, linear approximation to particle dynamics is no longer valid. Hamiltonian of averaged particle motion in general case is given by

$$H = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{p_\zeta^2}{2m\gamma^3} + qU_{ext} + q\frac{U_b}{\gamma^2}, \qquad (13)$$

where p_x, p_x p_ζ are momentum components of particle, oscillating around synchronous particle, U_{ext} is the potential of external field is and U_b is the space charge potential [4].

The general approach to finding a stationary, self-consistent beam distribution function is to represent it as a function of the constant of motion, $f = f(H)$, and then to solve Poisson's equation for the unknown space charge potential of the beam [4]. In Ref. [1] the self-consistent problem for a high-brightness beam in an RF field was solved assuming 6D beam distribution function is an exponential function of the Hamiltonian, $f = f_o \exp(-H/H_0)$. In particular, it was shown, that, to the first approximation, the self-consistent space charge potential of an intense beam is opposite to the external potential:

$$U_b = -\gamma^2 [1 - (\frac{\mu_t}{\mu_s})^2] U_{ext}. \qquad (14)$$

Equation (14) shows that the space charge field of a stationary bunch compensates for the external field in the beam interior. This phenomenon is well-known within the context of non-neutral plasma physics as Debye shielding. Equation (14) indicates that transverse and longitudinal oscillation frequencies in a self-consistent bunch are suppressed by space charge in the same proportion, which corresponds to a beam with equal space charge depression.

Performed analysis results in determination of a self-consistent bunch shape (see Fig. 4) [5]. It was shown that the shape of a matched bunch with constant particle density within the bunch is transformed from an ellipsoidal profile to a "separatrix"- type shape in real space when longitudinal beam size becomes comparable with separatrix size.

Figure 5 illustrates dynamics of a beam occupying 80% of separatrix, with initial ellipsoidal and self-consisted beam profile in a structure with $\varphi_s = -30°$, $\mu_{oz} = 29.8°$, $\mu_z = 16.4°$, $\mu_o = 76.7°$, $\mu_t = 35.4°$. As can be seen, beam with an initial ellipsoidal shape tends to change its profile to self-consisted one, while the shape of the matched beam is approximately conserved. Six-dimensional phase space volume growth is smaller in case of matched beam profile (see Fig. 6). Therefore, self-consistent beam profile provides additional specification on matched beam conditions together with solution of Eq. (6).

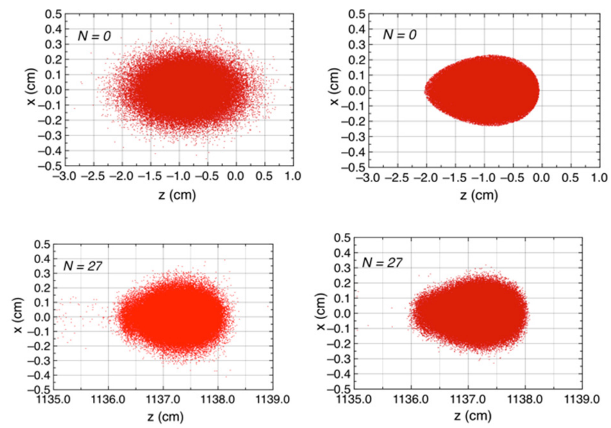

Figure 5: Dynamics of the beam: (left) with initial ellipsoidal shape, (right) with initial self-consistent profile, versus number of focusing periods N.

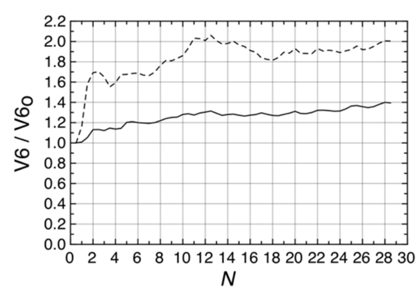

Figure 6: Six-dimensional phase space volume growth of the beam: (dotted line) with initial ellipsoidal shape, (solid line) with initial self-consistent profile.

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