MULTISCALE ANALYSIS OF RMS ENVELOPE DYNAMICS

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Abstract

We present applications of variational – wavelet approach to different forms of nonlinear (rational) rms envelope equations. We have the representation for beam bunch oscillations as a multiresolution (multiscales) expansion in the base of compactly supported wavelet bases.

1 INTRODUCTION

In this paper we consider the applications of a new numerical-analytical technique which is based on the methods of local nonlinear harmonic analysis or wavelet analysis to the nonlinear root-mean-square (rms) envelope dynamics [1]. Such approach may be useful in all models in which it is possible and reasonable to reduce all complicated problems related with statistical distributions to the problems described by systems of nonlinear ordinary/partial differential equations. In this paper we consider an approach based on the second moments of the distribution functions for the calculation of evolution of rms envelope of a beam. The rms envelope equations are the most useful for analysis of the beam self–forces (space–charge) effects and also allow to consider both transverse and longitudinal dynamics of space-charge-dominated relativistic high–brightness axisymmetric/asymmetric beams, which under short laser pulse–driven radio-frequency photoinjectors have fast transition from nonrelativistic to relativistic regime [1]. Analysis of halo growth in beams, appeared as result of bunch oscillations in the particle-core model, also are based on three-dimensional envelope equations [2]. From the formal point of view we may consider rms envelope equations after straightforward transformations to standard Cauchy form as a system of nonlinear differential equations which are not more than rational (in dynamical variables). Because of rational type of nonlinearities we need to consider some extension of our results from [3]-[10], which are based on application of wavelet analysis technique to variational formulation of initial nonlinear problems. Wavelet analysis is a relatively novel set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and give for the general type of operators (differential, integral, pseudodifferential) in such bases the maximum sparse forms. Our approach in this paper is based on the generalization [11] of variational-wavelet approach from [3]-[10], which allows us to consider not only polynomial but rational type of nonlinearities.

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Our representation for solution has the following form

$$z(t) = z_N^{slow}(t) + \sum_{j\geq N} z_j(\omega_j t), \quad \omega_j \sim 2^j \qquad (1)$$

which corresponds to the full multiresolution expansion in all time scales. Formula (1) gives us expansion into a slow part z_N^{slow} and fast oscillating parts for arbitrary N. So, we may move from coarse scales of resolution to the finest one for obtaining more detailed information about our dynamical process. The first term in the RHS of equation (1) corresponds on the global level of function space decomposition to resolution space and the second one to detail space. In this way we give contribution to our full solution from each scale of resolution or each time scale. The same is correct for the contribution to power spectral density (energy spectrum): we can take into account contributions from each level/scale of resolution. In part 2 we describe the different forms of rms equations. In part 3 we present explicit analytical construction for solutions of rms equations from part 2, which are based on our variational formulation of initial dynamical problems and on multiresolution representation [11]. We give explicit representation for all dynamical variables in the base of compactly supported wavelets. Our solutions are parametrized by solutions of a number of reduced algebraical problems from which one is nonlinear with the same degree of nonlinearity and the rest are the linear problems which correspond to particular method of calculation of scalar products of functions from wavelet bases and their derivatives.

2 RMS EQUATIONS

Below we consider a number of different forms of RMS envelope equations, which are from the formal point of view not more than nonlinear differential equations with rational nonlinearities and variable coefficients. Let $f(x_1, x_2)$ be the distribution function which gives full information about noninteracting ensemble of beam particles regarding to trace space or transverse phase coordinates (x_1, x_2). Then we may extract the first nontrivial bit of 'dynamical information' from the second moments

$$\begin{aligned}
\sigma^2_{x_1} &= <x_1^2> = \int\int x_1^2 f(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 \\
\sigma^2_{x_2} &= <x_2^2> = \int\int x_2^2 f(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 \qquad (2)\\
\sigma^2_{x_1x_2} &= < x_1x_2 > = \int\int x_1x_2 f(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2
\end{aligned}$$

RMS emittance ellipse is given by $\varepsilon^2_{x,rms} = < x_1^2 > < x_2^2 > - < x_1x_2 >^2$. Expressions for twiss parameters are also based on the second moments.

We will consider the following particular cases of rms envelope equations, which described evolution of the moments (1) ([1],[2] for full designation): for asymmetric beams we have the system of two envelope equations of the second order for σ_{x_1} and σ_{x_2}:

$$\begin{aligned}&\sigma''_{x_1}+\sigma'_{x_1}\frac{\gamma'}{\gamma}+\Omega^2_{x_1}\left(\frac{\gamma'}{\gamma}\right)^2\sigma_{x_1}= \\ &I/(I_0(\sigma_{x_1}+\sigma_{x_2})\gamma^3)+\varepsilon^2_{nx_1}/\sigma^3_{x_1}\gamma^2, \\ &\sigma''_{x_2}+\sigma'_{x_2}\frac{\gamma'}{\gamma}+\Omega^2_{x_2}\left(\frac{\gamma'}{\gamma}\right)^2\sigma_{x_2}= \\ &I/(I_0(\sigma_{x_1}+\sigma_{x_2})\gamma^3)+\varepsilon^2_{nx_2}/\sigma^3_{x_2}\gamma^2\end{aligned} \tag{3}$$

The envelope equation for an axisymmetric beam is a particular case of preceding equations.

Also we have related Lawson's equation for evolution of the rms envelope in the paraxial limit, which governs evolution of cylindrical symmetric envelope under external linear focusing channel of strenghts K_r:

$$\sigma''+\sigma'\left(\frac{\gamma'}{\beta^2\gamma}\right)+K_r\sigma=\frac{k_s}{\sigma\beta^3\gamma^3}+\frac{\varepsilon_n^2}{\sigma^3\beta^2\gamma^2}, \tag{4}$$

where $K_r \equiv -F_r/r\beta^2\gamma mc^2, \quad \beta\equiv\nu_b/c=\sqrt{1-\gamma^{-2}}$ According [2] we have the following form for envelope equations in the model of halo formation by bunch oscillations:

$$\begin{aligned}\ddot{X}+k_x^2(s)X-\frac{3K}{8}\frac{\xi_x}{YZ}-\frac{\varepsilon_x^2}{X^3}&=0,\\ \ddot{Y}+k_y^2(s)Y-\frac{3K}{8}\frac{\xi_y}{XZ}-\frac{\varepsilon_y^2}{Y^3}&=0,\\ \ddot{Z}+k_z^2(s)Z-\gamma^2\frac{3K}{8}\frac{\xi_z}{XY}-\frac{\varepsilon_z^2}{Z^3}&=0,\end{aligned} \tag{5}$$

where X(s), Y(s), Z(s) are bunch envelopes, $\xi_x,\xi_y,\ \xi_z=F(X,Y,Z)$.

After transformations to Cauchy form we can see that all this equations from the formal point of view are not more than ordinary differential equations with rational nonlinearities and variable coefficients (also,b we may consider regimes in which γ, γ' are not fixed functions/constants but satisfy some additional differential constraint/equations, but this case does not change our general approach).

3 RATIONAL DYNAMICS

Our problems may be formulated as the systems of ordinary differential equations

$$\begin{aligned}&Q_i(x)\frac{\mathrm{d}x_i}{\mathrm{d}t}=P_i(x,t), \quad x=(x_1,...,x_n), \\ &i=1,...,n, \quad \max_i deg\ P_i=p, \quad \max_i deg\ Q_i=q\end{aligned} \tag{6}$$

with fixed initial conditions $x_i(0)$, where P_i, Q_i are not more than polynomial functions of dynamical variables x_j and have arbitrary dependence of time. Because of time dilation we can consider only next time interval: $0\le t\le 1$. Let us consider a set of functions

$$\Phi_i(t)=x_i\frac{\mathrm{d}}{\mathrm{d}t}(Q_iy_i)+P_iy_i \tag{7}$$

and a set of functionals

$$F_i(x)=\int_0^1\Phi_i(t)dt-Q_ix_iy_i\mid_0^1, \tag{8}$$

where $y_i(t)$ $(y_i(0)=0)$ are dual (variational) variables. It is obvious that the initial system and the system

$$F_i(x)=0 \tag{9}$$

are equivalent. Of course, we consider such $Q_i(x)$ which do not lead to the singular problem with $Q_i(x)$, when $t=0$ or $t=1$, i.e. $Q_i(x(0)), Q_i(x(1))\neq\infty$.

Now we consider formal expansions for x_i, y_i:

$$x_i(t)=x_i(0)+\sum_k\lambda_i^k\varphi_k(t) \quad y_j(t)=\sum_r\eta_j^r\varphi_r(t), \tag{10}$$

where $\varphi_k(t)$ are useful basis functions of some functional space (L^2, L^p, Sobolev, etc) corresponding to concrete problem and because of initial conditions we need only $\varphi_k(0)=0$, $r=1,...,N, \quad i=1,...,n,$

$$\lambda=\{\lambda_i\}=\{\lambda_i^r\}=(\lambda_i^1,\lambda_i^2,...,\lambda_i^N), \tag{11}$$

where the lower index i corresponds to expansion of dynamical variable with index i, i.e. x_i and the upper index r corresponds to the numbers of terms in the expansion of dynamical variables in the formal series. Then we put (10) into the functional equations (9) and as result we have the following reduced algebraical system of equations on the set of unknown coefficients λ_i^k of expansions (10):

$$L(Q_{ij},\lambda,\alpha_I)=M(P_{ij},\lambda,\beta_J), \tag{12}$$

where operators L and M are algebraization of RHS and LHS of initial problem (6), where λ (11) are unknowns of reduced system of algebraical equations (RSAE)(12).

Q_{ij} are coefficients (with possible time dependence) of LHS of initial system of differential equations (6) and as consequence are coefficients of RSAE.

P_{ij} are coefficients (with possible time dependence) of RHS of initial system of differential equations (6) and as consequence are coefficients of RSAE. $I=(i_1,...,i_{q+2})$, $J=(j_1,...,j_{p+1})$ are multiindexes, by which are labelled α_I and β_I — other coefficients of RSAE (12):

$$\beta_J=\{\beta_{j_1...j_{p+1}}\}=\int\prod_{1\le j_k\le p+1}\varphi_{j_k}, \tag{13}$$

where p is the degree of polinomial operator P (6)

$$\alpha_I=\{\alpha_{i_1}...\alpha_{i_{q+2}}\}=\sum_{i_1,...,i_{q+2}}\int\varphi_{i_1}...\dot{\varphi}_{i_s}...\varphi_{i_{q+2}}, \tag{14}$$

where q is the degree of polynomial operator Q (6), $i_\ell = (1, ..., q+2)$, $\dot{\varphi_{i_s}} = \mathrm{d}\varphi_{i_s}/\mathrm{d}t$.

Now, when we solve RSAE (12) and determine unknown coefficients from formal expansion (10) we therefore obtain the solution of our initial problem. It should be noted if we consider only truncated expansion (10) with N terms then we have from (12) the system of $N \times n$ algebraical equations with degree $\ell = max\{p, q\}$ and the degree of this algebraical system coincides with degree of initial differential system. So, we have the solution of the initial nonlinear (rational) problem in the form

$$x_i(t) = x_i(0) + \sum_{k=1}^{N} \lambda_i^k X_k(t), \qquad (15)$$

where coefficients λ_i^k are roots of the corresponding reduced algebraical (polynomial) problem RSAE (12). Consequently, we have a parametrization of solution of initial problem by solution of reduced algebraical problem (12). The first main problem is a problem of computations of coefficients α_I (14), β_J (13) of reduced algebraical system. These problems may be explicitly solved in wavelet approach. The obtained solutions are given in the form (15), where $X_k(t)$ are basis functions and λ_k^i are roots of reduced system of equations. In our case $X_k(t)$ are obtained via multiresolution expansions and represented by compactly supported wavelets and λ_k^i are the roots of corresponding general polynomial system (12). Our constructions are based on multiresolution approach. Because affine group of translation and dilations is inside the approach, this method resembles the action of a microscope. We have contribution to final result from each scale of resolution from the whole infinite scale of spaces. More exactly, the closed subspace $V_j (j \in \mathbf{Z})$ corresponds to level j of resolution, or to scale j. We consider a multiresolution analysis of $L^2(\mathbf{R}^n)$ (of course, we may consider any different functional space) which is a sequence of increasing closed subspaces V_j: $...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ...$ satisfying the following properties:

$$\bigcap_{j\in\mathbf{Z}} V_j = 0, \quad \overline{\bigcup_{j\in\mathbf{Z}}} V_j = L^2(\mathbf{R}^n),$$

So, on Fig.1 we present contributions to bunch oscillations from first 5 scales or levels of resolution. It should be noted that such representations (1), (15) for solutions of equations (3)-(5) give the best possible localization properties in corresponding phase space. This is especially important because our dynamical variables corresponds to moments of ensemble of beam particles.

In contrast with different approaches formulae (1), (15) do not use perturbation technique or linearization procedures and represent bunch oscillations via generalized nonlinear localized eigenmodes expansion.

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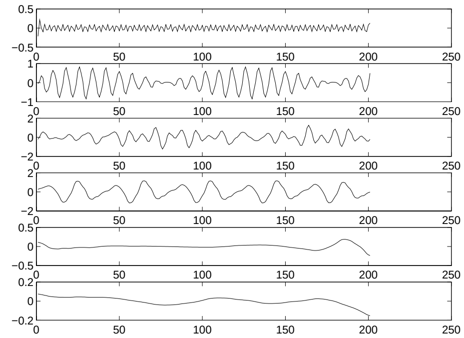

Figure 1: Contributions to bunch oscillations: from scale 2^1 to 2^5.

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