LINEAR THEORY OF IONIZATION COOLING AND EMITTANCE EXCHANGE*

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Abstract

The study of ionization cooling considered for muon colliders requires a full 6D treatment because of the need to exchange the longitudinal and transverse emittances. A general cooling channel consists of solenoids and quadrupoles for focusing, dipoles to generate dispersion, wedge absorbers for cooling and emittance exchange, and rf cavities for reacceleration. The quadrupole strengths can be adjusted so that the net focusing is cylindrically symmetric. The beam moments in such a system are completely specified in terms of five generalized emittances. We derive a set of coupled first-order differential equations describing the evolution of the generalized emittances due to the damping and excitation processes. The framework for lattice design is considered.

1 INTRODUCTION

In order to reduce both the transverse and longitudinal emittances of a muon beam for envisioned neutrino factories and muon colliders, 6D ionization cooling channels are being developed [1–4]. Promising designs consist of strong solenoids to provide transverse focusing, (gradient) dipoles to provide dispersion for emittance exchange, a low-frequency rf field to provide longitudinal acceleration and focusing, liquid hydrogen absorbers at minimum beta locations to provide ionization energy loss for transverse cooling, and wedged absorbers at maximum dispersion locations to provide momentum-dependent energy loss for longitudinal cooling. The linear Hamiltonian of such a focusing channel can be written as

$$\mathsf{H} = \frac{1}{2}\left(p_x^2+p_y^2\right)+\frac{1}{2}\kappa^2\left(x^2+y^2\right)-\kappa L_z \qquad (1)$$
$$-\frac{x\delta}{\rho}+\frac{x^2}{2\rho^2}+\frac{1}{2}g\left(x^2-y^2\right)+\frac{1}{2}\left(\frac{1}{\gamma_0^2}\,\delta^2+\mathrm{V}z^2\right)$$

where the spatial coordinates (x, y, z) and their corresponding canonical momenta (p_x, p_y, δ) are defined relative to a reference particle whose trajectory is a plane curve with radius of curvature $\rho(s)$ and follows the channel's layout. $L_z = xp_y - yp_x$ is the canonical angular momentum. The path length s along the reference orbit is used as the time variable, and $\delta = (p-p_0)/p_0$ is the relative longitudinal momentum deviation from the nominal momentum p_0. γ_0 is the Lorentz factor of the reference particle. The normalized on-axis solenoid field strength κ and quadrupole gradient g are given by $\kappa(s) = \frac{q}{2p_0}B_s(0,0,s)$ and $g(s) = \frac{q}{p_0}\frac{\partial B_y}{\partial x}$, where q is the muon's charge. $V(s)$ represents the longitudinal focusing from rf. For a gradient dipole with symmetric focusing, $1/\rho(s)^2 + g(s) = -g(s)$, and the total focusing strength becomes $K(s) = \kappa(s)^2 + 1/2\rho(s)^2$. Then the Hamiltonian

$$\mathsf{H} = \frac{1}{2}\left(p_x^2+p_y^2\right)+\frac{1}{2}K(s)\left(x^2+y^2\right)-\kappa(s)L_z \qquad (2)$$
$$-\frac{x\delta}{\rho(s)}+\frac{1}{2}\left[\frac{1}{\gamma_0^2}\,\delta^2+\mathrm{V}(s)z^2\right].$$

This Hamiltonian applies to linac-like single-pass channels as well as ring-like multi-pass channels that are under consideration. In any case, the underlying physics is 6D ionization cooling in a (quasi-) periodic channel.

To understand the basic beam dynamics of 6D ionization cooling and to establish a theoretical framework for the design of cooling channels, beam-moment equations have been developed in several papers for ionization cooling over the past several years [5–10]. Cooling dynamics described in the next section are based on Ref. [10]. There are 21 different second moments for a 6D phase space. In general, they are formidable to treat analytically. However, since we are mainly interested in cooling of a matched beam (i.e., it has equilibrium Gaussian distribution of the focusing channel) and the damping and excitations are small perturbations to the Hamiltonian motion, the moment equations can be reduced to evolution of beam-envelope functions characterizing the shape of the phase-space distribution and evolution of beam emittances characterizing the distribution density. The envelope functions are dominated by the strong Hamiltonian forces, and emittance evolution is determined by the small dissipative and diffusive forces. Beam evolution near equilibrium has been well treated in the context of radiation damping in electron storage rings [11]. The general formalism can be applied to the ionization cooling as well. In this short report, we briefly outline the theory of emittance evolution in section 2 and envelope-function design in section 3.

2 EMITTANCE EVOLUTION

In a cooling channel, the equation of motion using path-length s as the time variable is of the form

$$\frac{dX}{ds} = JHX + \left.\frac{dX}{ds}\right|_{\mathrm{M}}. \qquad (3)$$

Here, the first term on the right-hand side is the Hamiltonian part of the motion, where J is the simplectic ma-

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trix whose elements are the Poisson brackets of the phase-space variables, and H is the symmetric matrix associated with the above Hamiltonian H via $\mathsf{H} = X^T H X/2$. The last term in Eq. (3) represents the interaction with materials giving rise to weak dissipation and diffusion. It is of the form

$$\left.\frac{dX}{ds}\right|_{\rm M} = \left.\frac{dX}{ds}\right|_{\rm M,D} + \Xi = AX + \Xi. \qquad (4)$$

Here $(dX/ds)|_{\rm M,D}$ is the dissipative part of the interaction with material, A is the dissipation matrix, and Ξ represents the stochastic excitations discussed later in the moment equations. The dissipative part of the equation of motion is given by

$$\left.\frac{dx}{ds}\right|_{\rm M,D} = \left.\frac{dy}{ds}\right|_{\rm M,D} = \left.\frac{dz}{ds}\right|_{\rm M,D} = 0, \qquad (5)$$

$$\left.\frac{dp_x}{ds}\right|_{\rm M,D} = -\eta\left(p_x + \kappa y\right), \qquad (6)$$

$$\left.\frac{dp_y}{ds}\right|_{\rm M,D} = -\eta\left(p_y - \kappa x\right), \qquad (7)$$

$$\left.\frac{d\delta}{ds}\right|_{\rm M,D} = -(\partial_\delta\eta)\delta - (\partial_x\eta)x - (\partial_y\eta)y. \qquad (8)$$

Here $\eta = \frac{1}{pv}\frac{dE}{ds}$ is a positive quantity characterizing the average force due to ionization energy loss for a muon of momentum p and velocity v. The terms $(p_x + \kappa y)$ and $(p_y - \kappa x)$ are, respectively, the x and y components of the kinetic momentum. The wedged absorbers are treated as having uniform thickness with density depending linearly on the transverse coordinates. To linear order, the wedge absorber is characterized by $\partial_x\eta$ and $\partial_y\eta$, and the energy dependence of ionization energy loss is given by $\partial_\delta\eta$.

From the equation of motion, the moment equation reads

$$\frac{d\Sigma}{ds} = (JH + A_D)\Sigma + \Sigma\,(JH + A_D)^T + B. \qquad (9)$$

Here the quadratic beam-moment matrix $\Sigma = \langle XX^T\rangle$, the diagonal matrix $B = {\rm diag}(0, \chi, 0, \chi, 0, \chi_\delta)$ arising from the stochastic excitations represented by Ξ in Eq. (4). There are two different sources of excitations: multiple scattering characterized by the projected mean-square angular deviation per unit length $\chi = \left(\frac{13.6\,{\rm MeV}}{pv}\right)^2 \frac{1}{L_{\rm rad}}$, where $L_{\rm rad}$ is the radiation length of the absorbers, and energy straggling characterized by the mean-square relative energy deviation per unit length χ_δ.

To solve the cooling dynamics contained in Eqs. (3, 9), we first solve the Hamiltonian part that preserves the emittances and then compute the emittance evolution due to dissipation and diffusion. The Hamiltonian, Eq. (2), can be decoupled to a simple form by two canonical transformations: a rotation to the Larmor frame (rotating with the angle $\phi(s) = \int_0^s \kappa(\bar{s})d\bar{s}$) that decouples the two transverse degrees of freedom, and the dispersion transformation

$$\tilde{x} = \tilde{x}_\beta + \tilde{D}_x\delta\,, \quad \tilde{p}_x = \tilde{p}_{x_\beta} + \tilde{D}'_x\delta, \quad (x \leftrightarrow y) \qquad (10)$$

$$z = \hat{z} - \tilde{D}'_x\tilde{x} + \tilde{D}_x\tilde{p}_x - \tilde{D}'_y\tilde{y} + \tilde{D}_y\tilde{p}_y\,, \quad \delta = \hat{\delta} \qquad (11)$$

that decouples the transverse and longitudinal motions, provided that the dispersions D_x and D_y are zero in rf cavities and satisfy the equations

$$\tilde{D}''_x + K\tilde{D}_x = \frac{\cos\phi}{\rho}\,, \quad \tilde{D}''_y + K\tilde{D}_y = \frac{\sin\phi}{\rho}. \qquad (12)$$

Here the symbol ˜ indicates quantities in the Larmor frame and a prime indicates differentiation with respect to s. In terms of the betatron motion x_β and y_β and synchrotron motion $\hat{z}$, the new Hamiltonian simplifies to

$$\tilde{\mathsf{H}}_\beta = \frac{1}{2}\left(\tilde{p}^2_{x_\beta} + \tilde{p}^2_{y_\beta}\right) + \frac{1}{2}K\left(\tilde{x}^2_\beta + \tilde{y}^2_\beta\right) + \frac{1}{2}\left(I\delta^2 + {\rm V}\hat{z}^2\right), \qquad (13)$$

where $I(s) = \frac{1}{\gamma_0^2} - \frac{\tilde{D}_x\cos[\phi(s)]}{\rho(s)} - \frac{\tilde{D}_y\sin[\phi(s)]}{\rho(s)}$.

The Hamiltonian $\tilde{\mathsf{H}}_\beta$ has five linearly-independent quadratic invariants

$$I_x = \gamma_T\, x^2_\beta + 2\alpha_T\, x_\beta p_{x_\beta} + \beta_T\, p^2_{x_\beta}, \qquad (14)$$

$$I_y = \gamma_T\, y^2_\beta + 2\alpha_T\, y_\beta p_{y_\beta} + \beta_T\, p^2_{y_\beta}, \qquad (15)$$

$$I_z = \gamma_L\, \hat{z}^2 + 2\alpha_L\, \hat{z}\delta + \beta_L\, \delta^2, \qquad (16)$$

$$I_{xy} = \gamma_T\, x_\beta y_\beta + 2\alpha_T \frac{x_\beta p_{y_\beta} + y_\beta p_{x_\beta}}{2} + \beta_T\, p_{x_\beta}p_{y_\beta}, \qquad (17)$$

$$L_z = x_\beta p_{y_\beta} - y_\beta p_{x_\beta}. \qquad (18)$$

Here the envelope functions, γ_T, etc., are the periodic solution of the following familiar equations

$$\beta'_T = -2\alpha_T\,,\ \alpha'_T = K\beta_T - \gamma_T\,,\ \gamma_T = \frac{1+\alpha_T^2}{\beta_T} \qquad (19)$$

and

$$\beta'_L = -2I\alpha_T\,,\ \alpha'_L = V\beta_T - I\gamma_T\,,\ \gamma_L = \frac{1+\alpha_L^2}{\beta_L}. \qquad (20)$$

Averaged over the phase space, these five single-particle invariants lead to five beam invariants that are usually called beam emittances:

$$\epsilon_i = \frac{1}{2}\langle I_i\rangle, \quad i \in \{x, y, z, xy, L\}. \qquad (21)$$

Using emittances and invariants, the normalized equilibrium distribution can be written as

$$\rho(X) = \frac{1}{(2\pi)^3\epsilon_{\rm 6D}}\, e^{-\frac{\epsilon_y I_x + \epsilon_x I_y - 2\epsilon_{xy}I_{xy} - 2\epsilon_L L_z}{2(\epsilon_x\epsilon_y - \epsilon^2_{xy} - \epsilon^2_L)} - \frac{I_z}{2\epsilon_z}}, \qquad (22)$$

where the 6D emittance is

$$\epsilon_{\rm 6D} = (\epsilon_x\epsilon_y - \epsilon^2_{xy} - \epsilon^2_L)\,\epsilon_z. \qquad (23)$$

In a focusing channel without absorbers, the invariant emittances and the lattice functions $\beta_{T,L}$, $\alpha_{T,L}$, etc. determine the matched beam through the equilibrium phase-space distribution, Eq. (22).

We now address the effect due to interaction with material. Since the interaction is a weak perturbation to the Hamiltonian system, the beam phase-space evolution still follows the above equilibrium distribution but the emittances will slowly approach certain equilibrium values determined by the balance between ionization cooling and stochastic heating. The s-derivatives of the emittances can be computed by inserting the material part of the equation of motion into the derivative of Eq. (21). The stochastic contributions can be derived from Eq. (9). The results are

$$\epsilon_s' = -(\eta - ec_-)\epsilon_s + ec_+\epsilon_a + es_+\epsilon_{xy} + b\,\epsilon_L + \chi_s, \quad (24)$$
$$\epsilon_a' = -(\eta - ec_-)\epsilon_a + ec_+\epsilon_s + \chi_a, \quad (25)$$
$$\epsilon_{xy}' = -(\eta - ec_-)\epsilon_{xy} + es_+\epsilon_s + \chi_{xy}, \quad (26)$$
$$\epsilon_L' = -(\eta - ec_-)\epsilon_L + b\,\epsilon_s + \chi_L, \quad (27)$$
$$\epsilon_z' = -(\partial_\delta \eta + 2ec_-)\epsilon_z + \chi_z, \quad (28)$$

where ϵ_s and ϵ_a are the symmetric and asymmetric emittances $(\epsilon_x \pm \epsilon_y)/2$, $e = |\vec{D}| \cdot |\partial\vec{\eta}|/2$ is half of the maximum exchange rate through dispersions and wedges, $c_\pm = \cos(\theta_D \pm \theta_W)$ and $s_\pm = \sin(\theta_D \pm \theta_W)$ with θ_D and θ_W being the orientations of the dispersion vector and the wedges, and $b = \eta\kappa\beta_T + \alpha_T e s_- + \beta_T e' s_-'$ with $e' = |\vec{D}'| \cdot |\partial\vec{\eta}|/2$ and $s_-' = \sin(\theta_{D'} - \theta_W)$. The excitation terms are

$$\chi_s = \frac{1}{2}\beta_T\chi + \frac{1}{2}\mathcal{H}_s\,\chi_\delta, \quad (29)$$
$$\chi_a = \frac{1}{2}\mathcal{H}_a\,\chi_\delta, \quad (30)$$
$$\chi_z = \frac{1}{2}\beta_L\chi_\delta + \frac{1}{2}\gamma_L(D_x^2 + D_y^2)\chi, \quad (31)$$
$$\chi_{xy} = \frac{1}{2}\mathcal{H}_{xy}\,\chi_\delta, \quad (32)$$
$$\chi_L = \frac{1}{2}\mathcal{H}_L\chi_\delta. \quad (33)$$

Here the $\mathcal{H}$ functions are defined similarly as Eqs. (14-18) but phase-space variables are replaced with dispersion functions. For example, as in radiation damping theory, $\mathcal{H}_x = \gamma_T D_x^2 + 2\alpha_T D_x D_x' + \beta_T D_x'^2$. These heating terms arise from stochastic contributions to the beam invariants.

Note that the emittance exchange is accomplished by trading the damping rate ec_- between the transverse and longitudinal degrees of freedom. Without excitations,

$$\frac{d\,\epsilon_{6\mathrm{D}}}{ds} = -\left(2\eta + \partial_\delta\eta\right)\epsilon_{6\mathrm{D}}. \quad (34)$$

Therefore the total 6D damping rate is independent of the emittance exchange. This is equivalent to the Robinson theorem for radiation damping [12].

3 LATTICE-FUNCTION DESIGN

The emittance evolution, Eq. (24-28), shows that simultaneously cooling (negative damping coefficients) in both transverse and longitudinal phase space can be achieved through emittance exchange. Cooling behavior and equilibrium emittances can be computed when the lattice functions and absorbers are specified. However, designing a lattice with the desired lattice functions is a challenging task. Rather than blindly relying on simulations, Eqs. (19, 20, 12) can facilitate the lattice-function design. Even though nonlinear effects complicate the design process, the linear theory should still provide a good guidance in the early design stage since few machines work in a situation where their linear behavior performs badly.

In addition to numerical methods, analytical formulas for beta function and orbit stability are derived in Refs. [13, 14]. The focusing properties are determined by the Hills equation $\tilde{x}_\beta'' + K(s)\tilde{x}_\beta = 0$. For a periodic solenoidal channel, the field varies continuously with period L. It is natural to use the Fourier coefficients $\{\vartheta_n\}$ of the normalized focusing strength function $\vartheta(\varsigma) = \left(\frac{L}{\pi}\right)^2 K(\frac{L}{\pi}\varsigma)$ to characterize the solenoid field. Here $\varsigma = \pi\frac{s}{L}$ is the normalized position. The beta function can be calculated with

$$\beta(s) = \frac{L}{\pi}\frac{\sin(\sqrt{\vartheta_0}\pi)}{\sqrt{\vartheta_0}\sin\mu}\left[1 + \sum_{n=1}^{\infty}\frac{\Re[\vartheta_n e^{i2n\pi s/L}]}{n^2 - \vartheta_0} + \cdots\right]. \quad (35)$$

Here μ is the one-period phase advance that can be calculated via $\cos\mu = \Delta/2$, and Δ is the trace of the one-period transfer matrix that can be calculated with

$$\Delta = 2\cos(\sqrt{\vartheta_0}\pi) + \frac{\pi\sin\sqrt{\vartheta_0}\pi}{2\sqrt{\vartheta_0}}\sum_{n=1}^{\infty}\frac{|\vartheta_n|^2}{\vartheta_0 - n^2} + \cdots. \quad (36)$$

The orbit stability can be determined by the well-known criteria $|\Delta| < 2$. Higher-order expressions of β and Δ are available in Ref. [13]. Using these formulas, one can quickly estimate the basic properties of a solenoidal channel from the Fourier coefficients of its focusing function. More important than computing the values, insight can be gained from these analytical expressions.

Beyond the linear lattice design, ionization cooling channels need to confront severe nonlinearity due to the strong focusing required and compactness of the channel. On this front, not much has been done except simulations.

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