# HIGH FREQUENCY LINEAR ACCELERATOR: A PERTURBATIVE STUDY OF THE TRANSMISSION MATRIX OF A TWO PORT DEVICE CHAIN

A. D'Elia*, V.G. Vaccaro, DSF, Università di Napoli "Federico II" and INFN - M.R. Masullo#, INFN, Napoli, Italy - D. Davino, BNL, Upton - L. De Menna, S. Falco, DIEL, Napoli, Italy

*Abstract*

We study the high frequency behavior of a Side Coupled Linac (SCL) formed by a large number of accelerating cells. The SCL has as basic bricks copper plates (tiles) containing on one side half an accelerating cell and, on the reverse side, half a coupling one. The electromagnetic behavior of a single tile can be represented by a two-port device made of two lumped resonant series circuits coupled by a mutual inductance. The overall SCL behavior is so described by a two-port device chain. The transmission matrix of this chain is studied. Our goal is to study the overall behavior of an SCL structure in order to find out tools useful for the design, the analysis, the diagnostics and the correction of the single cells. In this paper, we consider two-port devices which are similar but exhibit slight differences, essentially due to fabrication errors which produce deviations from cell parameter nominal values. The chain transmission matrix has been studied by means of a perturbation analysis of the circuit parameters. Results on the relevant parameters, resonance frequency of each mode and accelerating voltage in the whole structure are presented.

## 1 INTRODUCTION

The SCL's [1] look quite promising since they may deliver the highest accelerating gradient to low energy proton beams. Because of this feature, they are shorter and more compact than the usual linacs; as a consequence they are quite suitable for accelerating proton beams for deep tumour cancer therapy [2] (protontherapy) in hospital environment [3]. A 3GHz SCL accelerator prototype for low energy protons (60MeV) has been recently designed, built and successfully tested [4]-[6]. Furthermore the excellent results obtained with the mentioned prototype enlarged the application area of these accelerators towards lower energies or higher frequencies (more compact realizations).

The fabrication tolerances play a crucial role in the performances of these devices because of the extremely high frequency of the RF feeders. In fact, the fabrication errors produce deviations from the design nominal values of the most relevant parameters.

Even if we are dealing with devices working in the Gigahertz range, lumped circuit representation very well suits the SCL behavior [7]. We resort to the transmission matrix representation of the two-port device and we investigate on the overall behavior of the SCL. In the case of N identical two-port device chain, the whole system exhibits N resonant frequencies, each characterized by its own mode (phase advance). These frequencies are given by simple analytical formulas [7]. In reality the situation is more intricate: after machining the tiles are unequal and the lumped parameter values of the device exhibit slight differences from tile to tile; furthermore these parameters, after assembling and brazing the tiles, exhibit additional random errors. At this stage it is impossible to make direct measurements of the single two port device, the only measurement allowed being on the whole system.

The transmission matrix of a chain of similar, but slightly different, two-port devices is studied here. The aim is to get information on the single cell parameters from the parameter measurements of the overall system (resonant frequencies, accelerating voltages, etc). We adopt the perturbation technique applied to transmission matrices.

## 2 THE MODEL

In general we describe the assembled tiles by means of a chain of two-port devices each denoted by the index $p$. We allow for the transmission matrix representation, so that the transmission matrix of the whole chain is just the ordered product of the transmission matrix of each cell $\mathbf{T}_p$[1]. The quantity $\mathbf{T}_p$ can be defined as:

$$\begin{pmatrix} V_p \\ I_p \end{pmatrix} = \begin{pmatrix} t_{11}^p & t_{12}^p \\ t_{21}^p & t_{22}^p \end{pmatrix} \begin{pmatrix} V_{p+1} \\ I_{p+1} \end{pmatrix} = \mathbf{T}_p \begin{pmatrix} V_{p+1} \\ I_{p+1} \end{pmatrix} \tag{1}$$

We suppose that each two-port device is composed by two lossy resonant series circuits [7] as shown in Fig. 1.

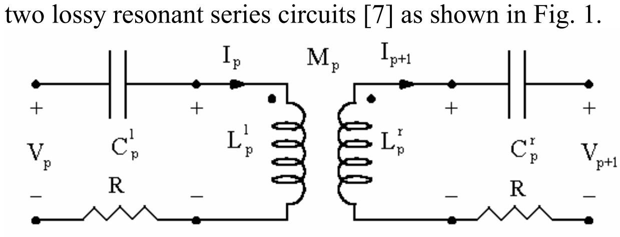


Fig.1. The generic two-port device representing two resonant series circuit coupled by a mutual inductance.

The two-port device parameters $L_p$, $C_p$ and $M_p$ differ for different values of $p$; in addition, as shown in Fig.1 where the indices $r$ and $l$ stand for right and left, in each two-

*DIEL, Univ. "Federico II" Napoli, Italy.

# masullo@na.infn.it.

port device the r.h.s. inductors and capacitors have different values indicating different fabrication errors. As a first approximation we may assume that the resistance does not depend on index *p*. It is straightforward to calculate the values of the transmission matrix elements of such a device.

Therefore we may define for each tile two resonant frequencies and a quality factor as

$$\omega_p = \frac{1}{\sqrt{L_p C_p}} \quad \text{and} \quad Q_p = \frac{1}{R}\sqrt{\frac{L_p}{C_p}} \qquad (2),$$

for the l.h.s and r.h.s. of the *p*-th two-port device.

A fundamental point of this model is to allow for finite losses, which are represented by the resistance R. This term is essential in order to evaluate the voltage behaviour at the accelerating gaps.

By definition the nominal resonant frequencies are calculated assuming that all tiles are equal:

$$L_p=L/2 \;;\; C_p=2C \;;\; M_p=M \;;\; Q_p = Q \qquad (3).$$

Resorting to formulas, which can be found in the literature [7,8,9] we get, without losses:

$$\Omega_s = \frac{\omega_0}{\sqrt{1+K\cos[(s-1)\pi/N]}} \text{, con } 1 \le s \le N+1 \qquad (4),$$

where the coupling coefficient $K$ and the angular frequency $\omega_0$ are defined as :

$$K = \frac{2M}{L} \quad \text{and} \quad \omega_0 = \frac{1}{\sqrt{LC}} \qquad (5).$$

The quantity $(s-1)\pi/N$ can be seen as the phase advance of the field along the chain and it is a characteristic of each mode. We are particularly interested in the so-called $\pi/2$ mode ($s=1+N/2$). In fact, because of its stability with respect to the errors, it is chosen for SCL operation.

The phase advance $\pi/2$ of this mode is such that any second cavity is empty of em-energy. The empty cavities are called coupling cavities (c.c.), the other ones are the accelerating cavities (a.c.).

Another useful formula is the impedance term of the transmission matrix of the whole system where all tile parameters have the nominal values :

$$(\mathbf{T}_{TOT})_{12} = \left(\mathbf{T}^N\right)_{12} \qquad (6).$$

It is convenient to have an equation in a factorised form in which we do not neglect the dissipative term:

$$\left(\mathbf{T}^N\right)_{12} = -j\omega M 2^{N-1}\prod_{i=0}^{N}\left(t_{11} - \cos\frac{i\pi}{N}\right) \qquad (7),$$

where

$$t_{11} = \frac{1}{K}\left(1 - \frac{\omega_0^2}{\omega^2} + \frac{\omega_0}{j\omega Q}\right) \qquad (8).$$

## 3 PERTURBATIVE MODEL

In this section we want to find the tools for the optimisation procedure using the system parameters which can be measured after the final assembly of the tiles.

In the following we will show how is possible to describe the real structure taking into account the errors due to fabrication tolerances by means of a perturbative technique. In the second section, we assume a finite quality factor (Q), while in the first one this parameter is kept infinite.

### *3.1 Perturbed resonant frequency*

Consider now a chain of N different two-port devices, where N is an even number. The overall transfer matrix is:

$$\mathbf{T}_{TOT} = \prod_{p=1}^{N}\mathbf{T}_p . \qquad (9),$$

where the product is ordered.

We look for the resonant frequencies without any restriction to the mode.

If the cell parameters are slightly different, the problem can be tackled by means of a perturbation technique. The equation to be solved is:

$$\left\{\prod_{p=1}^{N}[\mathbf{T}(\omega = \Omega_s + \Delta\Omega_s) + \Delta\mathbf{T}_p]\right\}_{12} = 0 \qquad (10)$$

where $\Delta\mathbf{T}_p$ is the perturbation in the matrix $\mathbf{T}_p$ and $\Omega_s$ is the unperturbed resonant frequency of the *s*- mode.

To a first order approximation, the above equation can be written:

$$\left\{\mathbf{T}^N(\Omega_s) + \left.\frac{d\mathbf{T}^N}{d\omega}\right|_{\omega=\Omega_s}\Delta\Omega_s + \sum_{p=1}^{N}\mathbf{T}^{p-1}\Delta\mathbf{T}_p\mathbf{T}^{N-p}\right\}_{12} = 0 \qquad (11)$$

From eq.(11), we can get the solution for the perturbed frequency :

$$\Delta\Omega_s = -\sum_{p=1}^{N}\left(\mathbf{T}^{p-1}\Delta\mathbf{T}_p\mathbf{T}^{N-p}\right)_{12} / \frac{d\mathbf{T}_{12}^N}{d\Omega_s} \qquad (12).$$

If we define

$$\Delta\omega_p = \frac{1}{2}\left(\Delta\omega_p^l + \Delta\omega_{p-1}^r\right) \qquad (13),$$

and assume

$$\Delta\omega_0^r = \Delta\omega_{N+1}^l = 0 \qquad (14),$$

after some algebra, we get the following expression for the perturbed frequency of the modes as a linear combination of the perturbed resonant frequencies of the cells:

$$\Delta\Omega_s\left(1 - K\cos\frac{(s-1)\pi}{N}\right)^{\frac{3}{2}} + \Delta\Omega_{N+2-s}\left(1 - K\cos\frac{(N+1-s)\pi}{N}\right)^{\frac{3}{2}} \equiv$$

$$2\Delta\overline{\Omega}_s = \frac{4}{N}\sum_{p=1}^{N+1}\Delta\omega_p\cos^2\frac{(p-1)(s-1)\pi}{N}; \quad s = 1 \ldots N+1 \qquad (15).$$

In the previous system (15) the equations of index *s* and *N+2-s*, are identical. As consequence, the number of the independent equations are:

$$E((N+1)/2) \qquad (16)$$

where the symbol *E(x)* means the first integer equal or larger than *x*. This property implies that the system (15) cannot be solved respect to $\Delta\omega_p$. However defining the new unknown

$$\Delta\overline{\omega}_p = \Delta\omega_p + \Delta\omega_{N+2-p} \quad (17),$$

equation (15) can be rearranged in the following form

$$\Delta\overline{\Omega}_s = \frac{2}{N}\sum_{p=1}^{E}\Delta\overline{\omega}_p \cos^2\frac{(p-1)(s-1)\pi}{N} \quad (18),$$

where $E$ stands for (16). In equation system (18) the number of unknowns ($\Delta\overline{\omega}_p$) is equal to the measurable quantities ($\Delta\overline{\Omega}_s$), and the determinant of system matrix **M** is finite; so that the system can be inverted. The inversion can be done analytically. It can be shown that:

$$\mathbf{M}^{-1} = \frac{4}{N}\mathbf{DMD} \quad (19)$$

where **D** is a diagonal matrix with $D_{ii}=1/2$ ($i=1,1+N/2$) and $D_{ii}=1$ otherwise.

It is interesting to analyse the case of the ($\pi$/2)-mode (case $s=1+N/2$). According to (15), the deviation of the mode frequency is:

$$\Delta\Omega_{1+N/2} \equiv \Delta\omega_0 = \frac{2}{N}\sum_{p=1}^{N+1}{}' \Delta\omega_p \quad (20),$$

where the prime index means the sum of only odd terms; this means that only the accelerating cells are involved in the detuning. We learn from (20) that it is not necessary to tune each single cell.

### *3.2 Feeding current*

Another interesting feature of the optimisation for an SCL is to maximise the feeding current as a function of the tuning. We consider an SCL operating on $\pi$/2 mode and with the feeding in the system central cell; this implies that cell number N is the quadruple of an integer number.

It is possible to show that the current $I_0$ and the feeding voltage $V_0$ are linked by

$$I_0 = V_0 \left(\prod_{p=1}^{N/2}\mathbf{T}_p\right)_{11}\left(\prod_{p=1+N/2}^{N}\mathbf{T}_p\right)_{22} \Bigg/ \left(\prod_{p=1}^{N}\mathbf{T}_p\right)_{12} \quad (21)$$

The denominator of the previous equation contains two terms: one unperturbed equal to eq.(7) and a perturbed one which has to be calculated without losses similarly to eq. (10) but for the $\pi$/2 mode case.

To a first order approximation we can neglect the resistive part and it is possible to show that the numerator is equal to one.

From these assumptions and after manipulations of the above formula, we obtain:

$$I_0 = \frac{V_0 Y_0}{\left(Q^{-1} - j\sum{}' \Delta\omega_i/\omega_0\right)} \quad (22),$$

where the characteristic admittance is: $Y_0 = 2\sqrt{C/L}$. The previous formula is most interesting because it shows a dependence of the current on the detuning errors. From the eq. (22) it is simple to get the accelerating voltage on the gaps multiplying the vector ($V_0$, $I_0$) by the appropriate transmission matrices. This multiplication will not introduce any additional resonance.

## 4 CONCLUSIONS

It is very important for a good design and operation of an high frequency Linac to have a good model which takes into account the real e.m. parameters of the structure (errors in the distributed parameters, resistive losses…).

In this paper we shown our perturbative model and we found out the formula to compute the mode frequency deviation as a function only of sum of the detunings of the symmetric accelerating cells. This is an important result because we learnt that at first order approximation the system is sensitive only to the sum of errors in symmetric cells. Particularly we carried out a simple formula for the $\pi$/2-mode which shows that for tuning it is sufficient to minimise the mean value frequency deviation.

We calculated for the same mode the feeding current taking into account the resistive losses. The formula is most interesting since it shows that the current maximisation is sensitive to the detuning mean value magnified by the quality factor Q.

On the base of these simple formulas, we can think to extend the feeding current expression to the other modes. Similarly we may calculate in a very simple way the accelerating gap voltage.

## 5 REFERENCES

[1] T. Wangler, *RF Linear Accelerators*, John Wiley & sons: New York, 1998, p. 102;

[2] Orecchia R, et al. "*Particle beam therapy (hadrontherapy): basis for interest and clinical experience*".; Eur J Cancer 1998 Mar;34(4):459-68;

[3] U. Amaldi and G. Tosi, "*Overview of the Hadrontherapy project and summary of this volume*", The Blue Book-The TERA project and the center for oncological hadrontherapy , July 1994;

[4] P. Berra et al., "*Study, Construction and Test of a 3 GHz Proton Linac-Booster (LIBO) for Cancer Therapy*", Proceedings of EPAC2000 Conference, Wien, 2000;

[5] B. Szeless and 14 co-authors, "*Successful High Power Test of a Proton Linac Booster (LIBO) Prototype for Handrontherapy*", PAC 2001, Chicago;

[6] C. De Martini et al, " *Beam Tests on a Proton Linac Booster for Hadrontherapy*" *Proceedings of EPAC2002 Conference*, Paris2002;

[7] D.E. Nagle, E.A. Knapp and B.C. Knapp, "*Coupled resonator model for standing wave accelerator tanks*", The Review of Scientific Instruments, (38)11, pp. 1583 - 1587, 1967;

[8] D. Davino et al., "A Perturbative Study of the Scattering Matrix of a Two-port Device Chain" Proceedings of the ISTET 2001, Linz, August 2001;

[9] V.G. Vaccaro et alii, "Perturbative techniques applied to a lumped circuit model of an SCL" in preparation.