MIXING OF REGULAR AND CHAOTIC ORBITS IN BEAMS*

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Abstract

Phase mixing of chaotic orbits exponentially distributes the orbits through their accessible phase space. This phenomenon, commonly called "chaotic mixing", stands in marked contrast to phase mixing of regular orbits which proceeds as a power law in time. It is inherently irreversible; hence, its associated e-folding time scale sets a condition on any process envisioned for emittance compensation. We numerically investigate phase mixing in the presence of space charge, distinguish between the evolution of regular and chaotic orbits, and discuss how phase mixing potentially influences macroscopic properties of high-brightness beams.

1 INTRODUCTION

We adopt the viewpoint that, under the influence of space charge, the evolution of beams, and of confined nonneutral plasmas in general, may be understood in terms of phase mixing of the constituent particle orbits. For example, linear Landau damping is merely phase mixing of regular orbits [1], a process by which initially neighboring orbits diverge secularly, i.e., as a power law in time [2]. A given space-charge potential may or may not support a population of globally chaotic orbits, i.e., orbits that wander over a large portion of their accessible phase space. Initially neighboring globally chaotic orbits fill their accessible phase space exponentially, a process known as "chaotic mixing" that was initially conceived in the astrophysical context of galactic dynamics [3,4]. When a substantial population of globally chaotic orbits exists, it dissipates correlations irreversibly. In beams the consequence is an irreversible emittance growth. Inasmuch as chaotic mixing is irreversible and acts exponentially, it is essential to identify conditions for its presence in beams, and to quantify the time scale of the associated dynamics.

2 THEORETICAL ESTIMATE

A semianalytic theory exists that relies on assumptions of ergodicity and a microcanonical distribution to estimate the largest Lyapunov exponents, i.e., the chaotic-mixing rates, in lower-dimensional, e.g., fully coarse-grained, time-independent Hamiltonian systems [5]. Chaos arises generically from a parametric instability that can be modeled by a stochastic-oscillator equation; linearized perturbations of a chaotic orbit satisfy a harmonic-oscillator equation with a randomly varying frequency. The underlying assumptions are, strictly speaking, invalid, yet the theory commonly yields estimates that are good to within a factor ~2 [6].

Applied to space-charge potentials, the theory yields an estimate of the chaotic-mixing rate λ as:

$$\lambda(\rho) \simeq (\kappa/3)^{1/2}\,[L^2(\rho) - 1]/L(\rho); \qquad (1)$$

in which

$$\kappa = \left(\omega_f^{\,2} - \omega_{po}^{\,2}\langle\nu\rangle\right)/2,$$
$$L(\rho) = \left\{T(\rho) + \left[1+T^{\,2}(\rho)\right]^{1/2}\right\}^{1/3},$$
$$T(\rho) = \left(3^{3/2}\pi\rho^2\right)/\left\{8\left[2(1+\rho)^{1/2} + \pi\rho\right]\right\},$$
$$\rho = \left[2(\langle\nu^2\rangle - \langle\nu\rangle^2)\right]^{1/2}/\left[(\omega_f/\omega_{po})^2 - \langle\nu\rangle\right];$$

$\omega_f = (\omega_x^{\,2} + \omega_y^{\,2} + \omega_z^{\,2})^{1/2}$ and ω_{po} refer to the external focusing frequency and the plasma frequency at the system's centroid, respectively, ν is the density normalized to the centroid density, and "$\langle q\rangle$" denotes a phase-space average of quantity q weighted by the microcanonical ensemble. In a system that is moderately out of equilibrium, one would expect to have $\rho \sim 1$ typically, for which $\lambda/f \sim 0.82$, with $f \equiv \kappa^{1/2}/(2\pi)$ representing the "dynamical frequency", i.e., the average orbital frequency. Thus, in such systems, the chaotic-mixing time scale is roughly one dynamical time. Thus, for one to be reasonably sure of its efficacy, a process of emittance compensation, i.e., removal of correlations within the beam, should be completed within a plasma period as measured from the source of the correlations.

3 NUMERICAL EXPERIMENTS

3.1 Equipartitioning

In a recent computational study using the 2-1/2 D version of the particle-in-cell code WARP, we discovered strong evidence that chaotic mixing is intimately connected with equipartitioning in beams [7]. This work concerned a highly space-charge-dominated, direct-current, cylindrical beam in which the initial momentum space reflected an anisotropic pressure such that $p_{xx}=2p_{yy}$. As the beam evolved, the pressure became increasingly isotropic on a rapid time scale. Though the relaxation time for two-body collisions in this beam corresponds to a propagation distance ~1 km, the beam equipartitioned in only ~5 m, followed by anisotropic pressure oscillations that largely damped by ~50 m. The underlying dynamics is manifestly collision*less*. The equipartitioning time scales were seen to correlate with the evolution of initially localized ensembles of particles. These ensembles expanded exponentially with an e-folding "time" ~2 m, which is about two plasma periods, and filled their accessible phase spaces in ~50 m. Moreover, plots of

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individual orbits appeared to reflect globally chaotic behaviour in keeping with the exponential dynamics. The beam parameters for this experiment were a plasma period of 1.14 m and x and y betatron focusing periods without space charge of 1.63 m. Inserting these numbers, along with ρ=1 and $\langle v \rangle$=1, into Eq. (1) yields an estimated e-folding (mixing) time ~2.0 m, in agreement with the simulation.

This first study comprises a form of "symmetry breaking", wherein the broken symmetry is in momentum space rather than configuration space. The beam thus begins in a nonequilibrium state, and it evolves toward a metaequilibrium in which the particle orbits have filled an invariant measure of phase space. The transient dynamics reflect an intricate, evolving network of space-charge waves that set up a complicated potential in which a substantial population of particle orbits becomes globally chaotic. By contrast, the symmetric, isotropic system establishes a potential that is strictly integrable, apart from granularity, in which the orbits are accordingly regular.

3.2 Five Beamlets in Smooth Transport Channel

A well-known experiment in accelerator physics is that of M. Reiser and collaborators [8] concerning the propagation of five beamlets in a periodic solenoidal transport channel. The beam is nonrelativistic and subject to considerable space-charge forces. The relaxation time via two-body collisions in this beam corresponds to a propagation distance ~1 km. Yet, regardless how well the beam was root-mean-square (rms) matched to the transport channel, the beamlets were seen to reappear only once, at a point ~1 m from the source. Their failure to reappear again would seem to reflect a collisionless process that, in effect, causes the particle orbits to lose memory of their initial conditions. Simulations with a particle-in-cell code well reproduced the measurements.

To explore how chaotic mixing influences the dynamics of such a manifestly nonequilibrium beam, we simulated the experiment using WARP. Our simulation differed from the experiment only in that we took the transport channel to impart a constant, linear external focusing force, whereas in the experiment the channel comprised a periodic solenoidal focusing lattice. Nonetheless, our simulation results correlate well with the measurements.

The strongly time-dependent space-charge potential drives a large population of globally chaotic orbits. Figure 2 on the next page illustrates how orbits of representative test particles evolve. The test particles interact with the potential but not with each other. One sees that typical ensembles that are initially localized in phase space grow exponentially to fill much of their respective accessible regions of phase space. Meanwhile the five beamlets lose their identity.

In this experiment, though chaotic orbits are easily found, it is difficult to separate the macroscopic influence of chaotic mixing from that of linear phase mixing of the five beamlets. Because the beamlets are large, they span a broad band of orbital frequencies in the initial potential. Accordingly, they smear through large regions of phase space and quickly overlap. One can be sure, however, that chaotic mixing is active over the bulk of phase space.

Analogous behaviour is seen in simulations of a rms-mismatched five-beamlet system, except now there is an additional phenomenon, namely, the formation of a prominent halo. Indications from the simulation are that the halo forms via parametric resonance with oscillations of the global potential as envisioned by Gluckstern [9]. Yet microscopic processes can stochastically convert core orbits to halo orbits and vice versa, thereby providing a mechanism for the production of "new halo" [10].

3.3 Chaos in Time-Independent Potentials

We now consider time-independent potentials in thermal equilibrium. The corresponding density profiles are constant near the bunch centroid, and at larger radii they drop to a low-density tail. In this “Debye region”, the net force on a particle is nonlinear, and one asks whether this force can support a substantial population of chaotic orbits. All spherically symmetric systems are integrable and support only regular orbits. In general, however, a system will be aspherical because the external focusing is generally anisotropic in a reference frame comoving with the beam.

A methodology for exploring these systems computationally is to integrate orbits that start from a very close distance in phase space, i.e., by placing them with zero initial velocity at nearby points in configuration space. Because the coarse-grained net force in these systems is conservative, the total particle energies E are conserved. The integration proceeds for about one dynamical time, at which point the Lyapunov exponent is calculated from the particle separations. The integration is then "renormalized" to bring the orbits close again, and the process is repeated until the Lyapunov exponents converge, which typically corresponds to a duration of ~200 dynamical times.

Upon expressing all lengths in units of the Debye length as measured at the bunch centroid, and all times as the product of ω_{po} with the real time t, one obtains the dimensionless potential-density pair

$$\Phi(\boldsymbol{x}) = (\Omega^2/2)[(b/a)^2 x^2 + y^2 + (b/c)^2 z^2] + \Phi_{sc}(\boldsymbol{x}), \qquad (2)$$
$$n(\boldsymbol{x}) = \exp[-\Phi(\boldsymbol{x})];$$

wherein Ω and (a,b,c) denote the strength and scale lengths, respectively, of the external focusing field, and $\Phi_{sc}(\boldsymbol{x})$ is the space-charge potential. Fig. 3 pertains to a triaxial configuration corresponding to Ω^2 = 1.0002/3; $(a/b)^2$ = 4/5, $(c/b)^2$ = 4/3. The results reflect statistics from samplings of ~2000 particle orbits that were started at zero velocity at various points in configuration space (corresponding to various total particle energies E). Plotted in Fig. 3 is the largest Lyapunov exponent, i.e., the chaotic-mixing rate of these orbits, normalized to the dynamical frequency. The theoretical result of Eq. (1) well matches the numerical result. Though this particular potential does admit chaotic orbits, they constitute only ~5% of all the sampled orbits. However, the fraction of

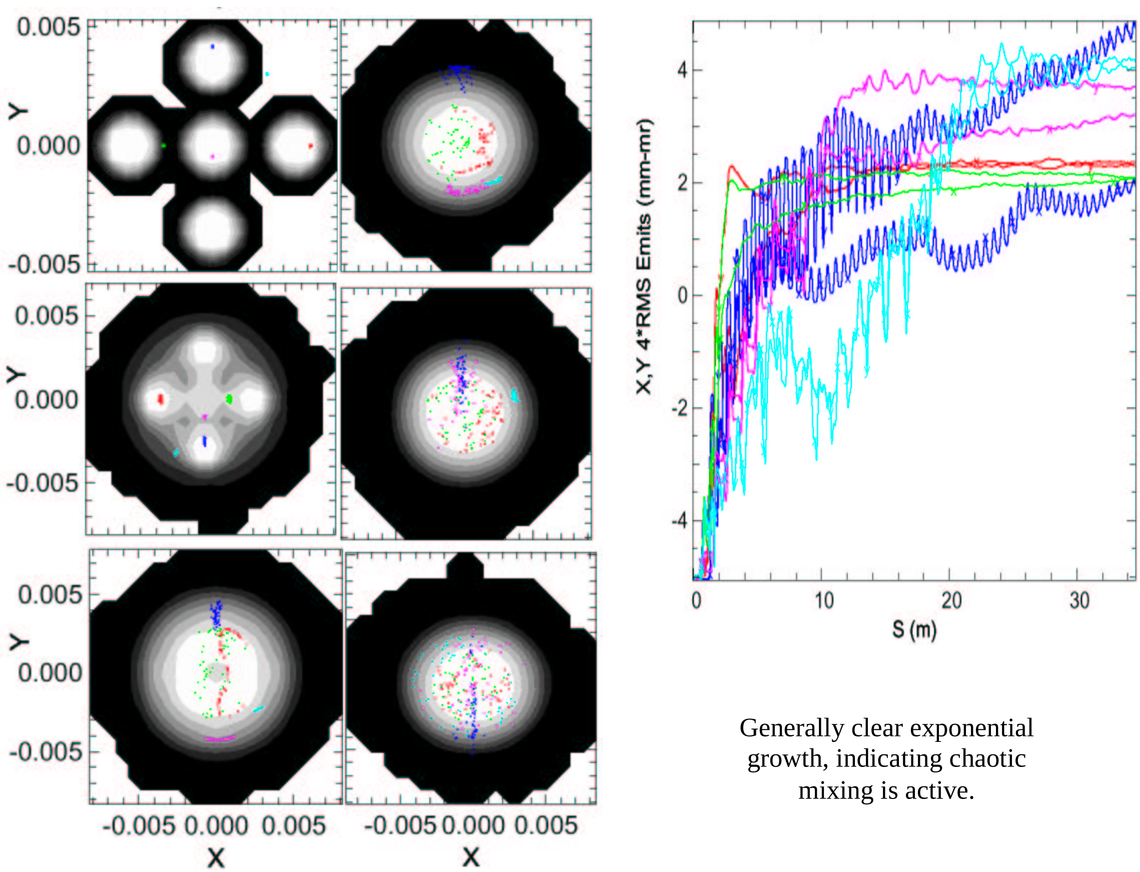

Figure 2: Evolution of five representative ensembles of test particles in the five-beamlet simulation. Beam parameters are: 5 keV energy, 44 mA current, 4.6 mm radius, and 64.8 μm full (90%) emittance. The left panel shows snapshots at (top-to-bottom left column) 0 m, 0.98 m, 2.88 m and (top-to-bottom right column) 5.24 m, 11.52 m, 31.68 m. The right panel shows the evolution of the natural logarithm of the *x* and *y* "emittance" moments of the ensembles.

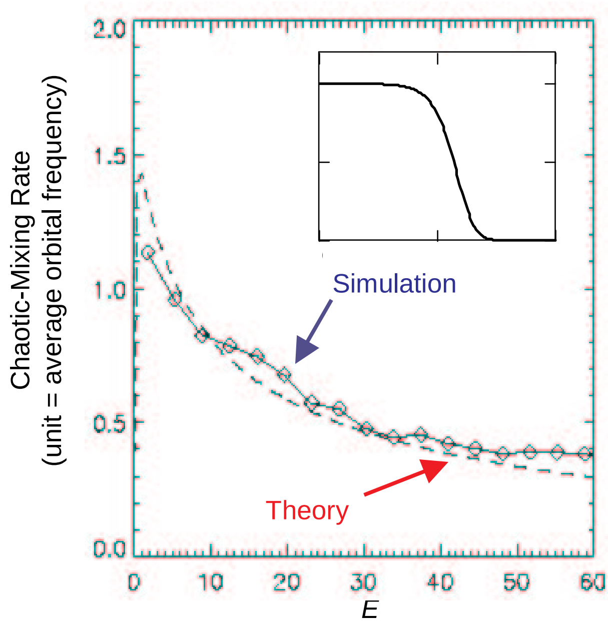

Figure 3. Chaotic-mixing rate versus total particle energy *E*. The inset shows the corresponding density profile along the y-axis with $x = z = 0$.

chaotic orbits depends on the geometry. We find over a substantial range of the parameter space it can be as high as a few tens of percent for orbits reaching to the Debye region. This is a significant result, in that density perturbations arising from irregularities in the external force will first appear in the Debye region, and the sizeable percentage of chaotic orbits will work toward irreversibly mixing these perturbations away.

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