L. Teng (Argonne National Laboratory) Linac phase oscillations.*

In a linear accelerator radial oscillations are usually strongly influenced by phase oscillations. It can be said however, that the phase oscillations are usually not strongly influenced by the radial oscillations, therefore, it is useful, in first approach to treat only uncoupled phase oscillations. This has been done in the following analysis. In this only the useful traveling wave component has been used. The reverse wave and all higher order components are being neglected as representing only perturbations on the phase oscillations. In one phase oscillation the effects of these perturbations average to zero.

Using now the equations of motion describing acceleration by the traveling wave component

$$E_{\text{longitudinal}} = \mathcal{E}_0 \sin \omega_0 \left\{ t - \int \frac{ds}{v_0(s)} \right)$$

one obtains with $\eta = \dfrac{p}{m_0 c}$

$$\frac{d\eta}{dt} = \frac{e\mathcal{E}_0}{m\ c} \sin \omega_0 \left(t - \int \frac{ds}{v_0(s)} \right)$$

It is useful here to use the coupled linear form:

$$\frac{d\eta}{ds} = \frac{e\mathcal{E}_0}{mc^2 \beta} \sin \phi \qquad \text{where } \phi = \omega_0 \left\{ t - \int \frac{ds}{v_0(s)} \right)$$

$$\frac{d\phi}{ds} = \frac{\omega_0}{c\ \beta} - \frac{\omega_0}{v_0(s)} \qquad \text{and } \omega_0 = \text{constant.}$$

*This material has been elaborated in the reports AGS-LA , LCT-1 and LCT-3.

With $\eta^2 = \gamma^2 - 1$ and $\beta = \frac{\eta}{\gamma}$ this is transformed to the following canonical form with γ and ϕ as variables.

$$\frac{d\phi}{ds} = \frac{\omega_0 \gamma}{c\sqrt{\gamma^2 - 1}} - \frac{\omega_0}{v_0(s)} = \frac{\partial H}{\partial \gamma}$$

$$\frac{d\gamma}{ds} = \frac{e\mathcal{E}_0}{mc^2} \sin\phi = -\frac{\partial H}{\partial \phi}$$

where

$$H = \frac{\omega_0}{c}\sqrt{\gamma^2 - 1} - \frac{\omega_0 \gamma}{v_0(s)} + \frac{e\mathcal{E}_0(s)}{mc^2} \cos\phi$$

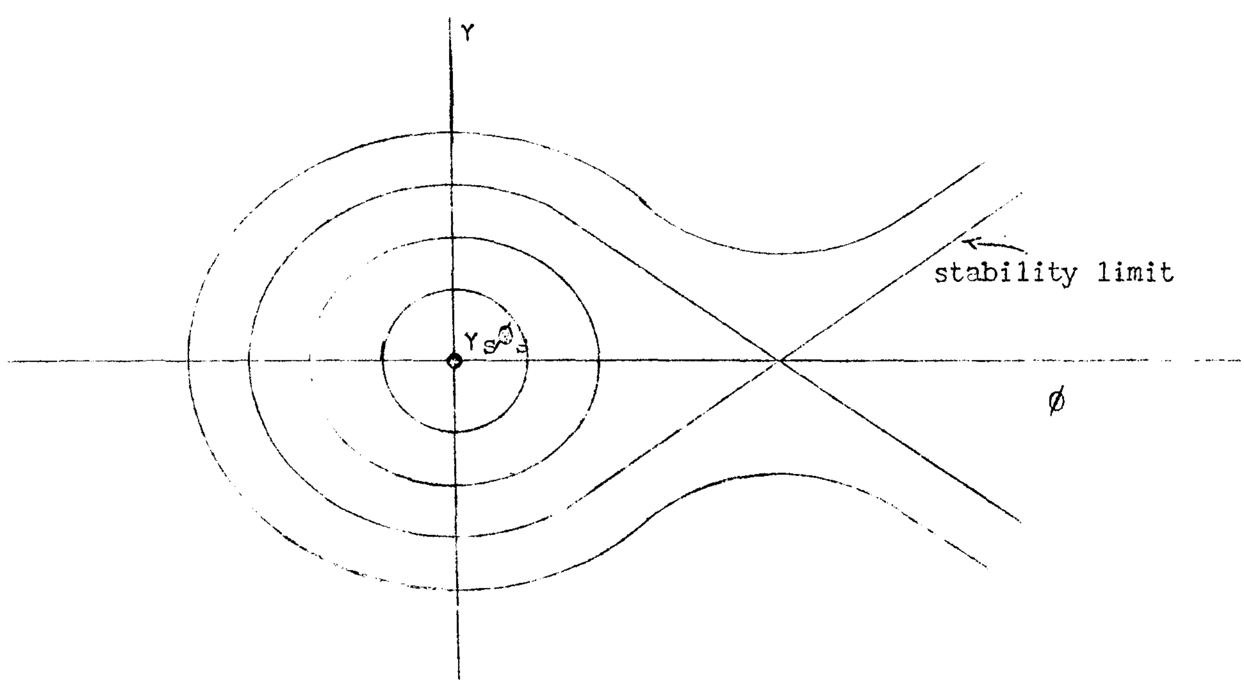
This is the Hamiltonian for pure uncoupled phase motion; if coupled to the radial motion more terms are used. It is assumed here that $\mathcal{E}_0(s)$ may be a function of s only. However, to solve

$$\frac{d\phi}{ds} = \frac{\partial H}{\partial \gamma}$$

$$\frac{d\gamma}{ds} = -\frac{\partial H}{\partial \phi}$$

adiabatically, it is necessary that $\mathcal{E}_0$ is independent of s or only slowly varying with s and $v_0(s)$ also is slowly varying with s.

The solutions are readily found for small amplitude oscillations by integrating twice. For larger oscillations the behaviour is non-linear and the first integrals, namely H=constant, take the form as shown below in a γ-ϕ plot.



For small oscillations assume

$$\gamma = \gamma_s(s) + \Delta\gamma$$

$$\phi = \phi_s(s) + \Delta\phi .$$

substitution in the original equations results in **

$$\frac{d\,\Delta\phi}{d\,s} = -\frac{\omega_o}{c\,\beta_s^{\,2}}\left(\frac{d\,\beta}{d\,\gamma}\right)_s \Delta\gamma - \left[\left(\frac{\omega_o}{v_o} + \frac{d\,\phi\,s}{d\,s} - \frac{\omega_o}{c\,\beta_s}\right)\right]$$

$$\frac{d\,\Delta\gamma}{ds} = \frac{e\mathcal{E}_o}{mc^2}\cos\phi_s\,(\Delta\phi) + \left[\left(\frac{e\mathcal{E}_o}{mc^2}\sin\phi_s - \frac{d\,\gamma_s}{d\,s}\right)\right]$$

It is possible to solve these equations by taking the forms in brackets to be equal to zero. This is satisfied if the constants are adjusted accordingly. That is, taking $v_o = c\beta_s$ results in $\frac{d\phi_s}{ds} = o$ or ϕ_s = constant

and $$\frac{d\,\gamma_s}{d\,s} = \frac{e\mathcal{E}_o}{mc^2}\sin\phi_s$$

Using now adiabatic approximations one can get the damping terms. Because in

$$\frac{d\,\Delta\phi}{ds} = -\frac{\omega_o}{c\beta_s^{\,2}}\left(\frac{d\beta}{d\,\gamma}\right)_s \Delta\gamma .$$

** Note:
J.P. Blewett suggested a combined form for these equations as follows:

$$\frac{d^2}{du^2}\,(\eta\,\Delta\phi) + \left(\frac{K}{\eta} - 1\right)\,(\eta\,\Delta\phi) = 0$$

where $u = \sinh^{-1}\eta$ and $K = \frac{\omega\, m_o c\,\cos\phi_s}{e\,\mathcal{E}\sin^2\phi_s}$

No approximations were being used in the transformation of the original equations into the above expression.

the coefficient of $\Delta\gamma$ is a function of s in the γ-ϕ plot the ellipses immediately around γ_s, ϕ_s are not closed but take the form of a spiral around γ_s, ϕ_s.

For this treatment to be valid it is necessary that the coefficients in the above equation vary slowly, actually for a linear accelerator $(\beta_s)^{-2}\left(\frac{d\beta}{d\gamma}\right)_s$ does not vary slowly, consequently the results are approximate only.

Assuming for the moment that the adiabatic approximation is sufficiently valid, then the damping in zero order approximation is given by:

$$\Delta\phi \cong \eta^{-3/4} \qquad \text{(holds also in the relativistic region)}$$

similar $\Delta\gamma \cong \eta^{3/4}$

and $\Omega \cong \sqrt{\dfrac{\omega_o}{c\eta_s^3}\dfrac{d\gamma_s}{ds}\cot\phi_s}$

where Ω is the angular frequency of the phase oscillation given by $\dfrac{2\pi}{\Omega} = \lambda$.

To the first order these terms become

$$\Delta\phi \cong \eta^{-3/4}\left(1 + \frac{G}{4}\right)$$

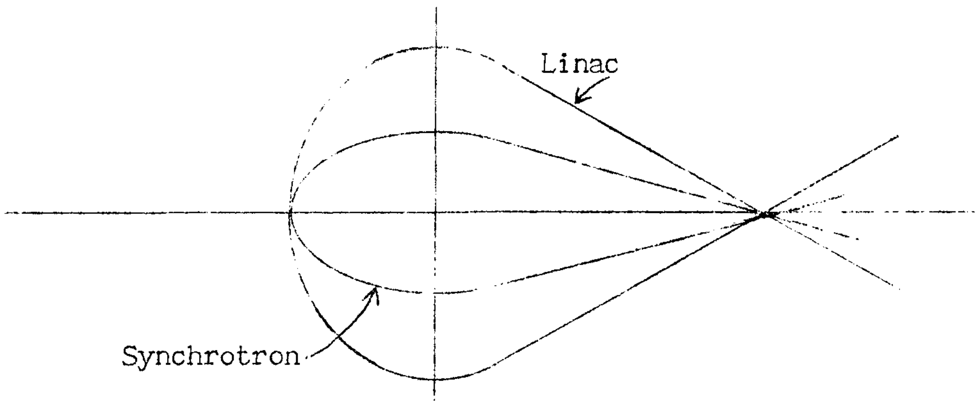
$$\Delta\gamma = \eta^{3/4}\left(1 - \frac{G}{4}\right)$$

$$\Omega \cong \sqrt{\frac{\omega_o}{c\eta_s^3}\frac{d\gamma_s}{ds}\cot\phi_s\,(1 - G)}$$

where G is given by

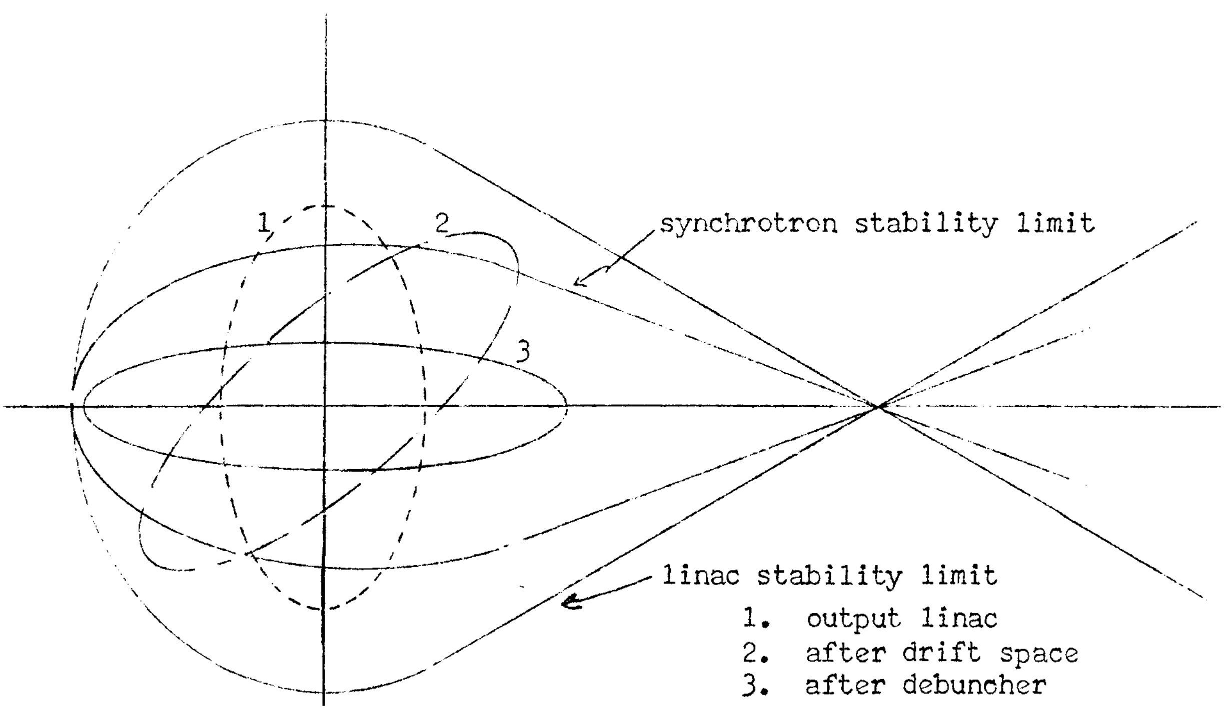
$$G = \frac{3}{4}\,\frac{\eta_s^{3/4}\dfrac{d}{ds}\left(\eta_s^{5/4}\dfrac{d\eta_s}{ds}\right)}{\dfrac{\omega_o}{c}\dfrac{d\gamma_s}{ds}\cos\phi_s}$$

The previous analysis with minor modifications applies also to a synchrotron ring. The stability limits for both linac and synchrotron ring are indicated below, assuming the same applied rf frequency.



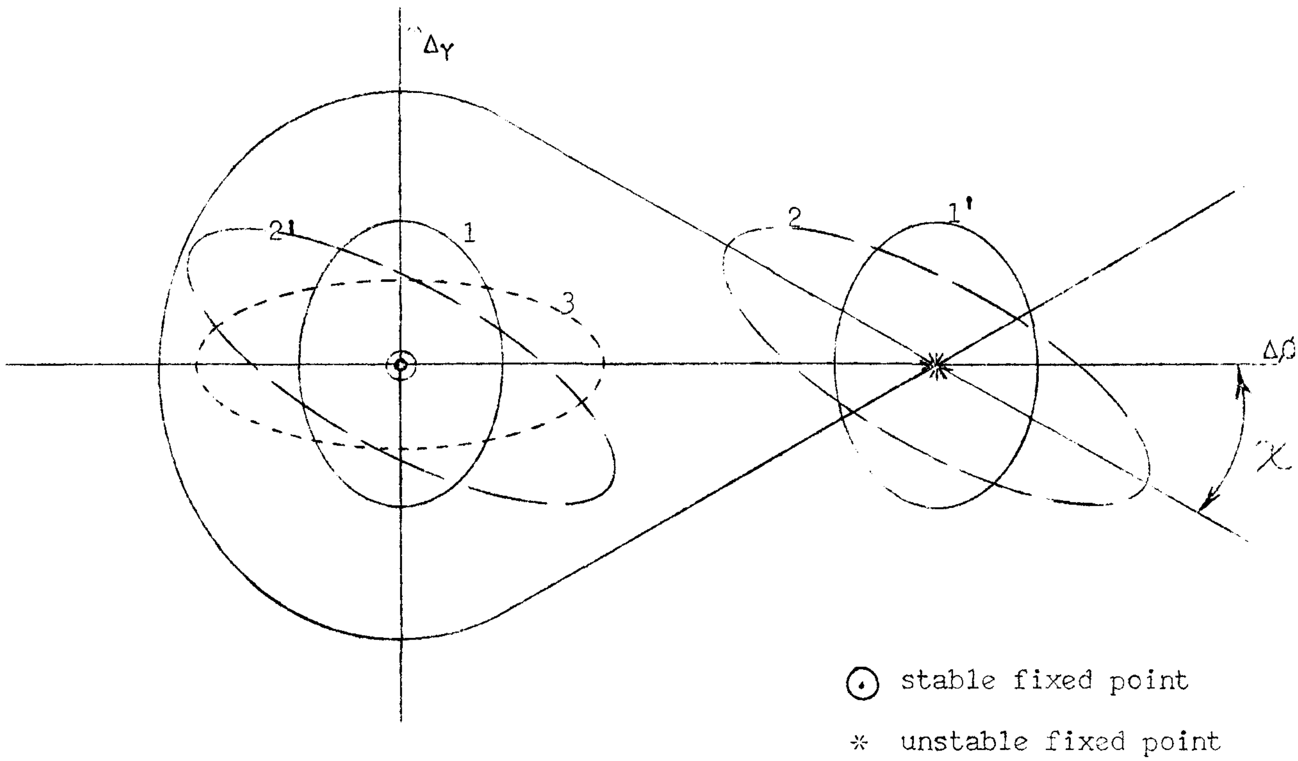
Consider now the problem of longitudinal phase-space matching of a linear accelerator with final particle energy of 10 Bev with let us say, a 1000 Bev synchrotron ring.

Using a conventional linac the output can be represented by curve 1 in the diagram below. Passing through a drift space the phase space area is indicated by curve 2, then using a debuncher curve 3 is obtained, which is suitable for acceptance in the synchrotron.



The problem is here that at 10 Bev particle energy the drift space needed is of the order of 10 miles. It would seem desirable to try to modify the linac in order to obtain directly curve 3 as shown above.

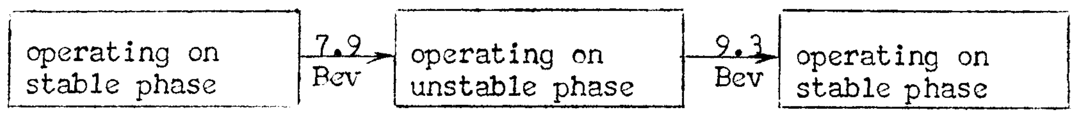
Consider again the $\Delta\gamma - \Delta\phi$ diagram.



Particles in the region near the stable point are being focussed, near the unstable point defocussed. Now it is possible in a linear accelerator to adjust the phase in certain sections such that a shift back and forth takes place from the area near the stable point to the area near the unstable point. Consequently one can alternately focus and defocus. By judicially adjusting the durations of the focussing and defocussing actions one can manipulate the ellipse to any desired shape.

The following example will illustrate this.

Linear accelerator sections



s(m)	T(Bev)	a	b	φ	
823	7.9	.026	.026	-	
967	9.3	.129	.042 10^{-1}	$-\frac{\pi}{4}$	here on unstable phase
1044	10	.122	.04 10^{-1}	0	here on stable phase

where a and b are the semi-axes of the ellipse and $\frac{d\gamma_s}{ds}$ was taken as 1 Bev per 350 feet.