F. Mills (MURA) Linac phase oscillations*

At MURA some concern existed as to the extent to which the phase motion was adiabatic. Therefore a Hamiltonian type analysis of the phase motion was developed. Then computations were performed in the energy region from 750 Kev to 200 Mev.

Using the conventional Hamiltonian expressions for particle dynamics:

$$\delta \int L dt = o \quad , \quad \delta \int (\Sigma p_i dq_i - H dz) = o.$$

one can derive all necessary expressions.

It is useful to formulate a new Hamiltonian in terms of the natural choice of variables in which E is conjugate to t and z is the independent variable.

i.e. $P_z = f\ (E,t,z)$

Instead of using approximations to the exact Hamiltonian to find the traveling wave component, one can average the equation of motion over a unit cell and derive an approximate Hamiltonian for the problem. These procedures yield identical results.

For pure axial motion (r = o) one obtains:

$$\left(\frac{dE}{dz}\right)_{average} = \frac{1}{L} \int e\, E_z(z') \sin(\omega t + \phi)\, dz'$$

This is to be evaluated along the particle path. Now assume that v is equal to its synchronous value v_s during its passage through the unit cell $\left(v_s = \frac{L\omega}{2\pi}\right)$.

* This work was done by F. Mills and D. Young

The $z = o$ value for the cell is defined by

$$\frac{1}{L} \int e\mathcal{E}_z(z') \sin \frac{2\pi z'}{L} \, dz' = o.$$

Then the energy gain is given by

$$\left(\frac{dE}{dz}\right)_{\text{average}} = T \mathcal{E}_o \sin \phi$$

$$\text{where } \mathcal{E}_o = \frac{1}{L} \int e\mathcal{E}_z(z') \, dz' \text{ and}$$

$$T = \frac{1}{\mathcal{E}_o L} \int e\mathcal{E}_z(z') \cos \frac{2\pi z'}{L} \, dz'$$

Analogous one can obtain:

$$\left(\frac{dt}{dz}\right)_{\text{average}} = \frac{1}{L} \int \frac{1}{v_z'} \, dz' = \frac{1}{v}$$

where v is referred to the unit cell.

From these equations the new Hamiltonian can be written as

$$\mathcal{H} = \frac{T\mathcal{E}_o}{\omega} \cos \phi + P_z(E)$$

It is clear from the above definitions that ϕ is the phase of the rf electric field when the particle crosses $z = o$ (in the unit cell) and therefore $t = \frac{\phi}{\omega}$.

The synchronous energy can be given as a function of z, from this the synchronous time can be derived. One can now make a series of canonical transformations to the synchronous energy and time coordinates,

i.e. given $E_s(z)$ one finds $v_s(z)$ and then

$$t_s(z) = \int_o^z \frac{1}{v_s(z')} \, dz'$$

The synchronous coordinate system e, τ is now defined by

$$e = E - E_s\ (z)$$

$$\tau = t - t_s\ (z)$$

This can be expressed as a sequence of two canonical transformations,

$$S_1 = E\ (\tau + t_s)$$

$$S_2 = \tau\,(e + E_s(z)\ )$$

The new Hamiltonian in the variables e and τ is now

$$K = \frac{T\mathcal{E}_o}{\omega} \cos \phi + \tau E_s'\ (z) + P_z\ (E_s + e) - \frac{E_s + e}{v_s(z)}$$

To obtain approximate solutions it is useful to expand $P_z\ (E_s + e)$ or $\cos \phi$.

It is clear that the result of this treatment is identical to that obtained by L. Teng (see VI) and in fact is identical in form to the phase oscillation theory for circular accelerators as developed by Symon and Sessler (MURA report 106)

From the above it is straightforward now to calculate the adiabatic invariant $J = \oint e\, d\tau$.

At this point a computer program was developed which calculated the phase motion step by step through a linear accelerator. At each step the value of the invariant is calculated using the above mentioned theory, to see whether or not J was indeed constant. For the particular case calculated TE_o = 2 Mev/m; sin ϕ_s = 0.9 ; initial energy 0.75 Mev; final energy 200 Mev.

It was found in general that J oscillated by perhaps as much as 20°/o but showed no general tendency to increase. Initial values for ϕ and E were chosen to blanket the expected stability region. The expected area in E, ϕ was calculated by the adiabatic damping rule and compared with the computed values. The expected energy spread was 1.12 Mev whereas the computed values had a spread of 1.05 Mev. A histogram of the final distribution is shown below.

ΔE	
202.24 - 202.34	x x x
.34 - .44	x x x
.44 - .54	x x x x x x x x
.54 - .64	x x x x x x x x x x x
.64 - .74	x x x x x x x x x x x
.74 - .84	x x x x x x x x x x x x x x x
.84 - .94	x x x x x x x x x x x x x
202.94 - 203.04	x x x x x x x x x x x x x x
.04 - .14	x x x x x x x x x x x x x
.14 - .24	x x x x x
.24 - .29	x x

$\frac{\Delta E}{E}$ for a 100°/o spread equals 0.52°/o.

" " 86°/o " " 0.35°/o.

" " 65°/o " " 0.25°/o.