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Linac focusing, based on work done at LRL, Berkeley*.

For a linac with particle energies in the Bev region, it might be useful instead of calculating the upper limit of the transverse phase space area at the output in the usual fashion, to extrapolate the experimental results obtained with the Brookhaven National Laboratory 50 Mev linac using the relation:

$$x\,\theta_x = \frac{P_o}{P}\,(x\,\theta_x)_r$$

where $(x\,\theta_x)_r$ indicates the phase space area experimentally obtained at 50 Mev.

$$(x\,\theta_x)_r \cong \pi . 10^{-3} \text{ cm rad.}$$

This results in the following values for higher linac output energies.

50 Mev	$x\,\theta_x \cong \pi . 10^{-3}$ cm rad.
630 Mev	$x\,\theta_x \cong (\pi/4) . 10^{-3}$ cm rad.
3 Bev	$x\,\theta_x \cong (\pi/12) . 10^{-3}$ cm rad.
10 Bev	$x\,\theta_x \cong (\pi/35) . 10^{-3}$ cm rad.

A linear accelerator, with final particle energy in the Bev region, would necessarily consist of several sections. It is therefore possible, instead of designing an accelerating structure on the basis of constant electric field amplitude and a phase velocity which increases continuously with distance (such that the concept of synchronous phase applies), to consider tne use of a uniform phase velocity, whereby then from section to section the phase velocity changes in discrete steps. In each section the particles will suffer a phase slip with consequent variation of the momentum spread. In order to maintain beam quality an upper limit has to be assigned to this. To get an impression of magnitude of the phase slip in each section the following analysis serves:

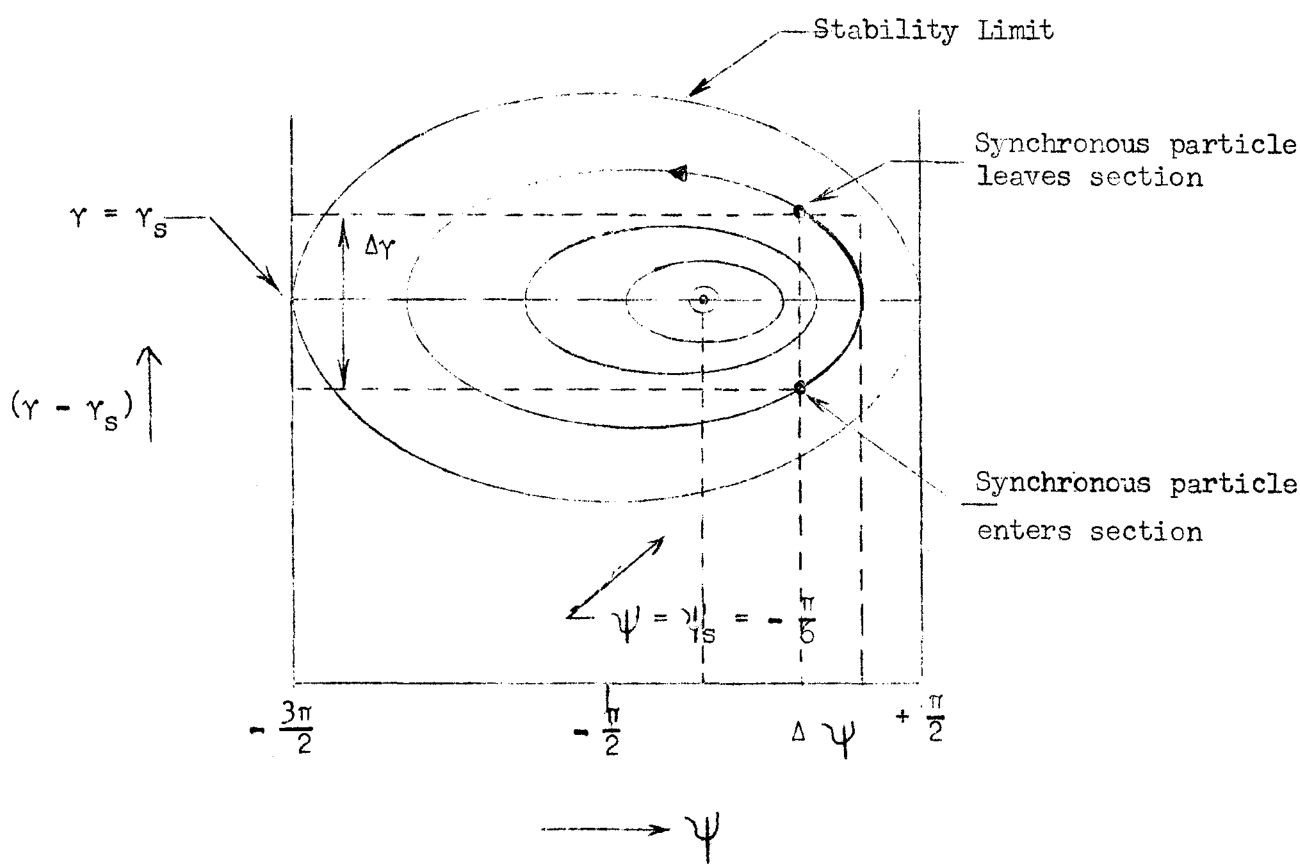
The axial motion of the particles is determined by (with the usual notation)

* Internal Reports Lawrence Radiation Laboratory LSI and LS3.

$$\frac{d\gamma}{dz} = \frac{e\,E_o(z)}{m_o c^2} \cos \psi$$

$$\frac{d\psi}{dz} = -\frac{2\pi}{\lambda} \frac{\gamma - \gamma_s}{\gamma_s^{\,3} \beta_s^{\,3}} \qquad \text{for } \gamma \cong \gamma_s$$

These equations describe the relationship between energy and phase deviations from synchronism. This behavior is shown below in a $(\gamma - \gamma_s)$ versus ψ diagram for constant phase velocity.



In each of the sections with uniform phase velocity, the behavior can be expressed by saying that the synchronous particle moves around on one of the curves within the stability limit. The extent of the phase slip in a particular section is indicated in the diagram. This can be split in half by allowing the particle to enter behind phase by 1/2 $(\Delta\psi)$ with respect to the phase for the section under consideration. The consequent increase in energy for this section is (see also diagram) $\Delta\gamma\, m_o c^2$. The $d\psi/dz$ equation can be integrated to obtain the numerical value for the maximum phase slip:

$$|\Delta\psi| = \frac{1}{4\gamma_s^{\,3}\,\beta_s^{\,3}}\left(\frac{\pi L\,\Delta\gamma}{\lambda}\right)$$

where $\Delta\gamma$ is given by $\gamma = \gamma_s + \frac{\Delta\gamma}{L}\, z$ and L is the section length.

Taking $L = 3$m.; $\lambda = 0.3$m.; $eE_o = 10$ Mev/m results in $\left(\frac{\pi L\Delta\gamma}{\lambda}\right) \cong 1.$

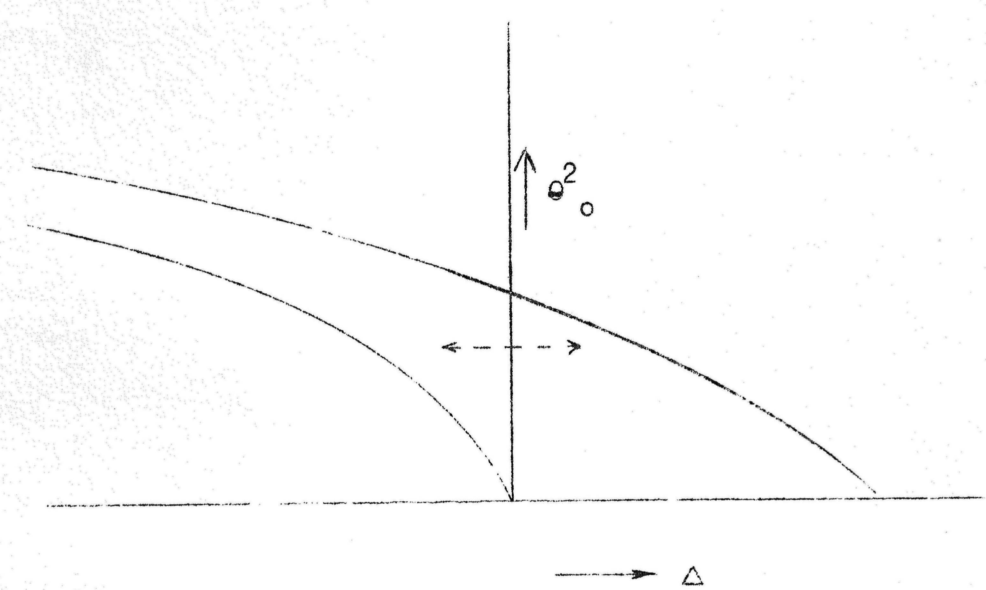
Therefore $|\Delta\psi| \cong 1/4 \dfrac{1}{\gamma_s^{\,3}\beta_s^{\,3}}$

Substituting values, one finds:

at 400 Mev	$\gamma\beta = 1$	$\Delta\psi = 15^\circ$
931 Mev	$\gamma\beta = \sqrt{3}$	$\Delta\psi = 3^\circ$
3 Bev	$\gamma\beta \cong 4$	$\Delta\psi \cong 0.2^\circ$
10 Bev	$\gamma\beta \cong 11$	$\Delta\psi \cong 0.01^\circ$

The magnitude of the phase slip is important mainly in that it gives rise to a transit time factor, lowering the rate of energy gain. In addition, higher order effects can lead to distortion of the phase oscillations and resulting deterioration of longitudinal beam quality. The Berkeley work suggests that these effects are unimportant above 400 Mev.

As a next step, one might consider omission of transverse focussing elements(Quadrupoles)in each particular section but placing these between sections. This whole procedure would not work at the low energy end because of the magnitude of the transverse excursion. Consider in this connection the diagram below.



where $\Delta = \frac{\text{Const. } \sin \Psi}{\beta}$ and Θ_o^2 = Const. $\beta B'$ where B' is the quadrupole field gradient. The solid lines indicate the stability limits. A particular ion with its values of β and Ψ can be represented in this diagram by a point which oscillates horizontally as shown. For low β values the B' values have to be high accordingly to remain within the stability limits. At higher β values the B' values can be lower and it would be possible to use quadrupole lenses only between the linac sections.

The linear accelerator consists now of a series of rf cavity structures in which phase restoring forces act, with drift spaces in between in which there are no phase forces. The effect is to change the phase oscillation wavelength

λ_ψ into a new wavelength Λ_ψ

$$\text{or} \quad \frac{\Lambda_\psi}{\lambda_\psi} = 1 + \delta .$$

A maximum value for the drift space ℓ is set by the desire to keep δ small. By using the criterium $\delta \leq 0.1$ it is possible to find an upper limit for ℓ, as follows:

The phase motion as expressed by ψ and $\frac{d\psi}{dz}$ in the center of one section can be related to that in the center of the preceding section by $M = M_{rf}\, M_d\, M_{rf}$

where

$$M_{rf} = \begin{pmatrix} \cos\frac{\pi L}{\lambda_\psi} & \frac{\lambda_\psi}{2\pi}\sin\frac{\pi L}{\lambda_\psi} \\ -\frac{2\pi}{\lambda_\psi}\sin\frac{\pi L}{\lambda_\psi} & \cos\frac{\pi L}{\lambda_\psi} \end{pmatrix} \quad \text{and} \quad M_d = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

Taking the product M and assuming $\ell/L \ll 1$ one finds:

$$\frac{\Lambda_\psi}{\lambda_\psi} \cong 1 + \frac{1}{2}\left(\frac{\ell}{L}\right)$$

Therefore with $\delta \leq 0.1$, the resulting maximum drift space length is $\ell \leq 60$ cm. Such a value of ℓ also leads to a 10% increase in radial amplitude for fixed angular divergence.

A **criterion** for stable transverse motion can be found by using the equation of motion for the transverse dimension r in the rf section:

$$\frac{d}{dt}\left(\gamma \frac{dr}{dt}\right) = \Omega^2 \gamma r \qquad \text{with } \Omega^2 = \left[\frac{-\pi e E \sin\psi_s}{m_o\, \beta \lambda \gamma^3}\right]$$

The matrix expression for the change in r and $\dot{r}$ for one section is then:

$$M_{rf} = \begin{pmatrix} \cosh\theta & \frac{1}{\Omega}\sinh\theta \\ \Omega\sinh\theta & \cosh\theta \end{pmatrix} \quad \text{with } \theta = \Omega\frac{L}{\beta c}$$

The motion in the drift space, neglecting the drift space length and taking the quadrupole as a thin lens, can now be given by:

$$M_q = \begin{pmatrix} 1 & 0 \\ -\delta_q & 1 \end{pmatrix}$$ where δ_q is the change in transverse velocity per unit displacement.

Multiplying now $M_q^{1/2}\, M_{rf} M_q^{1/2}$ one finds the matrix M for one rf section plus one drift space section

$$M = \begin{pmatrix} \cosh \Theta - \frac{\delta_q}{2\Omega} \sinh \Theta & \frac{1}{\Omega} \sinh \Theta \\ \Omega \sinh \Theta - \delta_q \cosh \Theta + \frac{\delta_q^2}{4\Omega} \sinh \Theta & \cosh \Theta - \frac{\delta_q}{2\Omega} \sinh \Theta \end{pmatrix}$$

This can be written as

$$M = \begin{pmatrix} \cos\mu & \alpha \sin\mu \\ -\frac{1}{\alpha} \sin\mu & \cos\mu \end{pmatrix} \quad \text{with } \cos\mu = \cosh \Theta - \frac{\delta}{2\Omega} \sin \Theta \ ; \text{ and } \alpha = \frac{\Theta}{\Omega\mu}$$

For a system with weak focussing μ and Θ will be small.

This gives $1 - \frac{\mu^2}{2} = 1 + \frac{\Theta^2}{2} - \frac{\delta_q}{2} \left(\frac{\Theta}{\Omega} \right)$ or

$$\delta_q = \frac{\Omega}{\Theta} \left(\mu^2 + \Theta^2 \right) \quad \text{and using } \Theta = \frac{\Omega L}{\beta c}$$

one finds $\delta_q = \frac{\beta c}{L} \left(\mu^2 + \frac{\Omega^2 L^2}{\beta^2 c^2} \right)$. For $\Omega L < \beta\mu c$ one can write

$$\delta_q \cong \frac{\beta c \mu^2}{L}$$

The optimum quadrupole strength will be found for minimum amplitude of oscillation, or considering the matrix M, for $\cos\mu = 0$.

$$\delta_q \text{ (optimum)} = 2\Omega \cosh \frac{\Omega L}{\beta c} \cong \frac{2\beta c}{L} \ .$$

With this an estimate for the quadrupole field gradients can be obtained. As will be seen however this leads to high fields and so μ will be taken as a design parameter.

Using a doublet with element length ℓ' and magnetic field gradient B' one finds

$$\frac{eB'}{m_o c^2} = \sqrt{\frac{3\beta\gamma^2\, \delta_q}{2\, \ell'^3 c}}$$

Using $\delta_q = \frac{\beta c \mu^2}{L}$

results in
$$\frac{\ell B'}{m_o c^2} = \left(\frac{3}{2 \lambda^3 L} \right)^{1/2} \mu\beta\gamma$$

Using a constant value for μ, B' is found to be proportional to $\beta\gamma$, increasing with proton energy. For $\mu \cong \frac{\pi}{2}$, corresponding to minimum phase oscillation amplitude, one finds for

$\beta\gamma = 0.63$ (200 Mev) $\quad B' \cong 2000$ gauss/cm

and for $\beta\gamma = 1$ (400 Mev) $\quad B' \cong 3000$ gauss/cm.

This is a fairly high field for quadrupoles and will have to be higher for higher $\beta\gamma$ values.

Since the quantity $\beta\gamma \left(\frac{x^2}{\alpha} + \dot{x}^2 \alpha \right)$ is an adiabatic invariant, one finds that the amplitude of transverse oscillation is given approximately by

$$A \cong \text{const} \sqrt{\frac{\alpha}{\beta\gamma}} \cong \frac{\text{const}}{\sqrt{\beta^2 \mu\gamma}}$$

It is clear that any tapering of μ to allow decrease of B' with γ makes A worse. However as in the electron accelerator, an unstable transverse "oscillation" does not lead to large displacements. Perhaps a suitable compromise would be to make μ vary as $(\beta\gamma)^{-1}$, keeping B' constant and letting A approach a constant value.

A crude estimate of the effect of a magnet misalignment has been made leading to an rms amplitude increase of

$$\langle \delta A \rangle \cong \sqrt{\frac{3}{4} \frac{L}{\ell}} \sqrt{N} \langle \Delta x \rangle$$

where ℓ is the length of a single magnet in the quadrupole pair, N is the number of magnets and $\langle \Delta x \rangle$ is the rms error in transverse alignment. This expression is independent of machine parameters, in the approximation $\Omega L < \beta\mu c$.

In a multiple section linear accelerator the effects of errors in individual sections will have an influence on the longitudinal phase motion of the particles and consequently on energy spread, which should be kept within acceptable limits for injection into the synchrotron. The influence of

a) phase error δ_ψ from one section to the next one

b) field gradient error ΔE from one section to the next one and

c) phase velocity error in a section $\Delta\beta_p$ in a section

can be expressed , taking the numerical values from LRL internal report LS3 for a 2 Bev linac, by

$$\overline{\Delta\psi} \cong 3\left\{\overline{\delta^2_\psi} + 3\,\overline{(\Delta E/E)^2} + 1000\,\overline{(\Delta\beta_p/\beta^2)^2}\right\}^{1/2}$$

From this, in order to keep the phase spread to the half width of the final bunch, it is required that $(\Delta E/E)_{rms} \cong 1$°/o , $(\delta_\psi)_{rms} \cong 1^\circ$ and $(\Delta\beta_p/\beta^2)_{rms} \cong 3.10^{-3}$.

It would seem not unreasonable to assume that some of the effects mentioned are coupled to another.

For example, changes due to temperature variations could affect the linac sections in such a way that cross terms would have to considered in the derivation of the above expression. This might make the errors either larger or smaller than with the neglect of these terms.