THE STEPPED PHASE VELOCITY LINEAR ACCELERATOR

H. B. Knowles
Yale University

As part of the Yale Design Study of Linear Accelerators the stepped phase velocity linac has been considered. The motivation for not changing β continuously but piecewise in an iris-loaded waveguide was originally an assumed high cost for the fabrication of many different kinds of sections in a very long proton accelerator. The work at Stanford, as described by R.B. Neal in the preceding report, suggests that a tapered β accelerator may not be very much more expensive than a stepped β machine. Further, if the design for a proton accelerator in the energy region above 200 Mev should be done for a frequency as low as 400 Mc/s, the waveguide cost will be mainly that for fabrication of the various sections without regard as to whether the phase velocity is tapered or stepped. In any case, some electron accelerators are operated at a constant phase velocity in the "bunching section" and resemble a stepped phase velocity accelerator in this area. For this reason, and because of the current speculation about the feasibility of constructing a meson accelerator, the following analysis has been done. First, the most general case will be considered and the approximations used will be indicated. Certain small terms can conveniently be dropped if the accelerated particles are protons, but should be retained if lighter particles are to be accelerated.

The Hamiltonian

We assume (1) constant gradient, and

(2) forward wave only.

These conditions are precisely realized in the Stanford M machine design, although the Yale design has been oriented to the π-mode iris-loaded standing-wave accelerator. The backward wave will induce small oscillatory perturbations on the phase and energy, which are not considered here, although it does not appear that the fundamental conclusions are invalidated by the backward wave. Analysis suggests that the effects of the backward wave are nullified because injection and ejection must each occur at an iris.

Consider now the equations for the axial motion

$$\frac{d\gamma}{dz} = \frac{eE}{mc^2} \cos \phi$$

$$\frac{d\phi}{dz} = \frac{2\pi}{\lambda} \left(\frac{1}{\bar{\beta}} - \frac{1}{\beta} \right) ,$$ with the usual notation,

and $\bar{\beta}$ = the constant phase velocity in the section in question and ϕ is the phase of the particle <u>forward</u> of the peak of the rf wave. This is the negative of the Ψ used by L. Smith*, because, as will be shown, the ϕ notation is useful in forming an optical analogy. Solving these equations (by the elimination of dz) and integrating, one obtains

$$\bar{\gamma}\, \gamma - \overline{\gamma\beta}\; \gamma\beta \;-\; q^2 \sin \phi = H\,(\gamma,\phi)$$

where

$$q^2 = \frac{eE\lambda}{2\pi mc^2}\; \bar{\gamma}\, \bar{\beta}$$

The q^2 value is an important parameter of the calculation. For proton accelerators and realizable values of E, $q^2 \simeq 10^{-4}$ and thus terms in q^2 (and in some cases of q) are neglected, while, for an electron accelerator, q is of the order of unity and many of the approximations given below are invalid**. The q and q^2 terms should be included for lighter particles.

---

*L. Smith, Internal Reports, LRL, LS-1 and LS-3.

**For mesons, q^2 is about 10^{-3} and the q value in this case is about midway between those for the proton and the electron.

H is the Hamiltonian for any one section. Defining a length parameter

$$\zeta = \frac{2\pi z}{\lambda \bar{\gamma} \bar{\beta}}$$

and differentiating, yields

$$\frac{\partial H}{\partial \phi} = - \frac{d\gamma}{d\zeta}$$

$$\frac{\partial H}{\partial \gamma} = + \frac{d\phi}{d\zeta}$$

such that γ has the character of a canonical momentum and ϕ that of its conjugate coordinate. Both Slater and Chu have formulated expressions quite similar to those above, although electrons were under consideration and $\gamma\beta$ was used rather than γ.

There are really three differential equations, but the third one (in dz) will be integrated when a canonical transform is done.

It is useful to define now

$$H = 1 - \sin^2 \bar{\phi} ,$$

by taking $\phi = \bar{\phi}$ when $\gamma = \bar{\gamma}$ ($\beta = \bar{\beta}$). $\bar{\phi}$ is the minimum phase angle a particle can have, as seen in Fig. 1, in which γ and ϕ are plotted for various values of $\bar{\phi}$, which represents H ($\bar{\phi}$). The quantity $\bar{\phi}$ characterizes an "orbit", that is, a single particle path. Clearly there is no stable phase angle in any constant phase velocity linac section, except for $\bar{\phi} = \pi/2$, and this is one in which no energy is gained.

The areas A_1 and A_2, shown in Fig. 1, are equal and show the nature of the distortion as a phase group moves through the accelerator.

Length and the Canonical Transformation

Obtaining z, or its representation ζ, involves a straightforward but tedious calculation. It is best explained in the following way: a distribution in γ and ϕ will become distorted as it moves through γ, ϕ space, yet, by Liouville's theorem, the density in canonical phase space must remain constant. It is desirable to transform the γ and ϕ into some other set of coordinates, say P and Q, in which the phase distribution remains undistorted. The Hamiltonian expresses libration in (γ,ϕ) space, so one can transform to (P,Q) space by means of the Hamilton-Jacobi principal function, specifically the one that is expressed in the old coordinate ϕ and the new momentum P; that is, W (ϕ,P). The new coordinates become

$P = H$ an energy-like coordinate

$Q = \zeta$ the time-like coordinate .

(It has been shown before that, in a linac, length replaces time in the dynamical relations.)

The "new coordinate" and "old momentum" are obtained from W by

$$\frac{\partial W}{\partial \phi} = \gamma$$

$$\frac{\partial W}{\partial P} = Q$$

W is obtained by first noting that

$$H + q^2 \sin \phi = P + q^2 \sin \phi = \bar{\gamma}\gamma - \overline{\gamma\beta}\,\gamma\beta$$

has the character of a "differential γ". Designating this as γ^*, then

$$\gamma^* = \bar{\gamma}\gamma - \overline{\gamma\beta}\,\gamma\beta = P + q^2 \sin \phi \,.$$

Similarly, there is a "differential momentum", given by

$$\gamma^*\beta^* = \bar{\gamma}\,\gamma\beta - \overline{\gamma\beta}\,\gamma = \sqrt{(P + q^2 \sin \phi) - 1} \,.$$

The notation

$$\rho = \frac{\gamma^*\beta^*}{\gamma\beta} = \bar{\gamma}\,(1 - \frac{\bar{\beta}}{\beta})$$

will be used in parts of the subsequent calculation. Note that both $\gamma^*\beta^*$ and ρ are positive quantities in the "upper limb" of the orbit $(\beta > \bar{\beta})$ and negative in the "lower limb" $(\beta < \bar{\beta})$. ρ and $\gamma^*\beta^*$ are small quantities (of the order of magnitude of q) in a proton accelerator. γ^* is almost unity (to order q^2), having values always greater than or equal to unity. To obtain the principal function W, consider

$$\frac{\partial W}{\partial \phi} = \gamma = \bar{\gamma}\,\gamma^* + \overline{\gamma\beta}\,\gamma^*\beta^* = \bar{\gamma}\,(P + q^2 \sin\phi) \pm \overline{\gamma\beta}\,\sqrt{(P + q^2 \sin\phi)^2 - 1}\,.$$

Therefore

$$W = \int_{\bar{\phi}}^{\phi} \gamma\,(P,\phi)\,d\phi\,,$$

and Q is obtained by

$$Q = \int_{\bar{\phi}}^{\phi} \frac{\partial\gamma(P,\phi)}{\partial P}\,d\phi = \int_{\bar{\phi}}^{\phi} \bar{\gamma}\,d\phi \pm \overline{\gamma\beta} \int_{\bar{\phi}}^{\phi} \frac{(P + q^2 \sin\phi)}{\sqrt{(P + q^2 \sin\phi)^2 - 1}}\,d\phi\,.$$

With the transformations

$$2\omega = \pi/2 - \phi$$

$$2\bar{\omega} = \pi/2 - \bar{\phi}$$

$$\sin\theta = \frac{\sqrt{\sin^2\bar{\omega} - \sin^2\omega}}{\sin\bar{\omega}\,\cos\omega}$$

and a table of integrals (see Peirce), one obtains

$$Q = \zeta = \frac{2\pi z}{\overline{\gamma\beta}\lambda} = \bar{\gamma}\,(\phi - \bar{\phi}) \pm \frac{\overline{\gamma\beta}}{q}\left[(1 + 2q^2 \cos^2\bar{\omega})\,F(k,\Psi) - 2q^2 \cos^2\bar{\omega}\,\Pi\,(k,\nu,\Psi)\right]$$

in which $F(k,\Psi)$ is an elliptic integral of the first kind, and $\Pi(k,\nu,\Psi)$ is a rare elliptic integral, which has apparently not been tabulated. Further,

$$k^2 = \sin^2\bar{\omega}\,(1 - q^2 \cos^2\bar{\omega}) \quad (\simeq \sin^2\bar{\omega} \text{ for proton accelerators})$$

$$\nu = -\sin^2\bar{\omega}$$

Ψ = the terminal value of θ at the ϕ (ω) value of interest.

For proton accelerators an adequate approximation is

$$\zeta \cong \bar{\gamma}\,(\phi - \bar{\phi}) \pm \frac{\overline{\gamma\beta}}{q}\,F(k,\Psi)\,.$$

The positive sign is used for the upper limb and the negative sign for the lower limb; ζ is defined as zero when $\phi = \bar{\phi}$.

This solves the third equation (in dz) subject to the condition $\zeta = 0$ when $\gamma = \bar{\gamma}$ and $\phi = \bar{\phi}$. The lines of equal ζ are asymmetrical because of the "invariant" $\bar{\gamma}$ $(\emptyset - \bar{\emptyset})$ term, and are more closely spaced for $\beta > \bar{\beta}$ than for $\beta < \bar{\beta}$ in any section. ζ will, in general, be less than zero for $\beta < \bar{\beta}$, and positive for $\beta > \bar{\beta}$.

When a phase group enters a section, the early particles (those with maximum $\emptyset$) will thus have a larger value of $|\zeta^-|$ than those arriving late, but a smaller ζ^+ value than the late particles. However, because a section is the same length for all particles in the group (see Fig. 2), the particle that has a larger than average value of injection phase $(\emptyset^-)$ will tend to have a <u>lower</u> than average value of ejection phase $(\emptyset^+)$, and vice versa. Consider now the phase group behavior in the (P,Q) coordinate system, shown in Fig. 3, where $P = (1 - q^2 \sin \bar{\emptyset})$ is the ordinate and $Q = \zeta$ the abscissa.

In this space, the phase group is undistorted and "flows" along the lines of P = constant, which correspond to the orbits.

<u>Transformation Matrix</u>

The behavior in P-Q space, which is a complete description, suggests that a distribution in γ^- and $\emptyset^-$ at injection can be converted to P and Q and returned to the γ, $\emptyset$ representation at ejection, i.e. to γ^+ and $\emptyset^+$, to form a matrix which describes the first order behavior of the phase group in the accelerating sections:

$$\begin{bmatrix} \delta\emptyset^+ \\ \delta\gamma^+ \end{bmatrix} = \begin{bmatrix} \frac{\partial\emptyset^+}{\partial\emptyset^-} & \frac{\partial\emptyset^+}{\partial\gamma^-} \\ \frac{\partial\gamma^+}{\partial\emptyset^-} & \frac{\partial\gamma^+}{\partial\gamma^-} \end{bmatrix} \begin{bmatrix} \delta\emptyset^- \\ \delta\gamma^- \end{bmatrix}$$

Here $\emptyset$ is analogous to x and γ to x', as in an optical matrix, because of the coordinate-momentum analogy. Also, if the first order behavior of a phase group in a field free drift space of length ℓ_n between the nth and (n+1)th section of a multisection accelerator is calculated, one obtains, to a first order

$$\begin{bmatrix} \delta\phi^-_{n+1} \\ \delta\gamma^-_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & + \dfrac{2\pi\ell_n}{\lambda(\gamma\beta)^3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta\phi^+_n \\ \delta\gamma^+_n \end{bmatrix} ,$$

provided that the frequency does not change at the end of this space. This matrix is completely analogous to a drift matrix in first order optical theory; this is the reason for reversing the phase angle notation.

The transformation matrix from entrance to exit in one section may be obtained from simultaneous differentiation of

$$P = \overline{\gamma}\,\gamma - \overline{\gamma\beta}\,\gamma\beta - q^2 \sin\phi$$

and

$$Q = \int_{\overline{\phi}}^{\phi} \left\{ \overline{\gamma} \pm \overline{\gamma\beta} \; \frac{(P + q^2 \sin\phi)}{\sqrt{(P + q^2 \sin\phi)^2 - 1}} \right\} d\phi \, .$$

The expression for Q is in P and ϕ, rather than γ and ϕ, and requires some tedious algebraic work. When completed, the transformation matrix at the injection end may be written as

$$\begin{bmatrix} \delta Q \\ \delta P \end{bmatrix} = \begin{bmatrix} M^- \end{bmatrix} \begin{bmatrix} \delta\phi^- \\ \delta\gamma^- \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\rho^-}\left[1 - \mu^-\chi^-\right] & \dfrac{\chi^-}{g} \\ - g\mu^- & \rho^- \end{bmatrix} \begin{bmatrix} \delta\phi^- \\ \delta\gamma^- \end{bmatrix}$$

and, at ejection as

$$\begin{bmatrix} \delta\phi^+ \\ \delta\gamma^+ \end{bmatrix} = \begin{bmatrix} M^+ \end{bmatrix} \begin{bmatrix} \delta Q \\ \delta P \end{bmatrix} = \begin{bmatrix} \rho^+ & - \dfrac{\chi^+}{g} \\ + g\mu^+ & \dfrac{1}{\rho^+}\left[1 - \mu^+\chi^+\right] \end{bmatrix} \begin{bmatrix} \delta Q \\ \delta P \end{bmatrix}$$

where ρ is as previously noted and

$$\mu^\pm = \frac{\cos\phi^\pm}{\cos\overline{\phi}} \qquad (\mu^\pm \text{ is always positive for } \left|\phi^\pm\right| \leq \tfrac{\pi}{2})$$

$$g = q^2 \cos\overline{\phi}$$

$$\chi^\pm = \left[\mu^\pm + \overline{\gamma}\,\rho^\pm\,(1 - \mu^\pm)\right] \qquad (\chi^\pm \cong \mu^\pm \text{ for small value of } q) \, .$$

Clearly, the behavior of a phase group is given by

$$\begin{bmatrix} \delta\phi^+ \\ \delta\gamma^+ \end{bmatrix} = \begin{bmatrix} M^+M^- \end{bmatrix} \begin{bmatrix} \delta\phi^- \\ \delta\gamma^- \end{bmatrix}$$

and the $\begin{bmatrix} M^+ M^- \end{bmatrix}$ matrix can be analyzed to yield

$$\begin{bmatrix} M^+M^- \end{bmatrix} = \begin{bmatrix} \rho^+ & 0 \\ g\mu^+ & \frac{1}{\rho^+} \end{bmatrix} \begin{bmatrix} 1 & -\frac{\chi^+}{g\rho^+} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & +\frac{\chi^-}{g\rho^-} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\rho^-} & 0 \\ -g\mu^- & \rho^- \end{bmatrix}$$

in which the first and last may be further separated and shown to represent image formation by a convex and a concave lens respectively, as follows:

$$\begin{bmatrix} \rho^+ & 0 \\ g\mu^+ & \frac{1}{\rho^+} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{(1-\rho^+)}{g\mu^+} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ g\mu^+ & 1 \end{bmatrix} \begin{bmatrix} 1 & +\frac{1}{g\mu^+}\frac{(1-\rho^+)}{\rho^+} \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{1}{\rho^-} & 0 \\ -g\mu^- & \rho^+ \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{g\mu^-}\frac{(1-\rho^-)}{\rho^-} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -g\mu^- & 1 \end{bmatrix} \begin{bmatrix} 1 & +\frac{(1-\rho^-)}{g\mu^-} \\ 0 & 1 \end{bmatrix}$$

that is, the injection end resembles a converging lens of "focal length"

$$f^- = (g\mu^-)^{-1} = (q^2 \cos \phi^-)$$

perceiving an object at a position

$$S_1^- = \frac{(1-\rho^-)}{g\mu^-} \quad \text{(recall that } \rho^- < 0)$$

just outside its focal length and forming an image at a conjugate point

$$S_2^- = -\frac{1}{g\mu^-}\frac{(1-\rho^-)}{\rho^-} .$$

This is followed by a <u>negative</u> drift space χ^-/ρ^-g long. The ejection end begins by resembling <u>another</u> negative drift space $-\chi^+/\rho^+g$ long, and is followed by the image formation of a diverging lens of (negative) focal length

$$f^+ = (g\mu^+)^{-1} = (q^2 \cos \phi^+)^{-1}$$

perceiving an object at distance

$$S_1^+ = \frac{1}{g\mu^+} \frac{(1-\rho^+)}{\rho^+} \qquad (\text{here } \rho^+ > 0)$$

and forming a "virtual" image at

$$S_2^+ = - \frac{(1-\rho^+)}{g\mu^+}$$

that is, <u>in front of</u> the lens and just <u>inside</u> the focal length, because ρ^+ is of the order of q.

The analogous optical system may be summarized as follows:

1. A primary object distance $= \frac{1-\rho^-}{g\mu^-} = \frac{1-\rho^-}{q^2 \cos \phi^+}$.
2. A converging lens of focal length $= (g\mu^-)^{-1}$.
3. A drift space of length $d = S_2^- + S_1^+ - \frac{1}{g}\left(\frac{\chi^+}{\rho^+} - \frac{\chi^-}{\rho^-}\right)$

$$d \simeq \frac{1}{q^2}\left\{ - \frac{1}{\rho^- \cos \phi^-}\left[1 - \left(\frac{\cos \phi^-}{\cos \bar{\phi}}\right)^2\right] + \frac{1}{\rho^+ \cos \phi^+}\left[1 - \left(\frac{\cos \phi^+}{\cos \bar{\phi}}\right)^2\right]\right\}$$

4. A diverging lens of focal length $(g\mu^+)^{-1}$.
5. Appearance (<u>not necessarily</u> of an image) at a distance $- \frac{(1-\rho^+)}{q^2 \cos \phi^+}$

   in <u>front</u> of the converging lens.

Clearly, the peculiar type of AG system shown in Fig. 4 is present. Depending on the choice of the parameters, focus may or may not occur at the output position, and it is equally uncertain that there will be phase compression.

The focusing is determined by the $(\partial\gamma^+/\partial\phi^-)$ term in the completed $\left[M^+M^-\right]$ matrix. When completely multiplied, this becomes

$$\frac{\partial\gamma^+}{\partial\phi^-} = g\left\{\frac{\mu^+}{\rho^-} - \frac{\mu^-}{\rho^+} - \frac{\mu^+\mu^-}{\rho^-\rho^+}(\rho^+\chi^- - \rho^-\chi^+)\right\} ,$$

analogous to the formula for combination of two lenses of focal lengths f_1 and f_2 separated by a distance a, in which

$$\frac{1}{f_1} = -\frac{g\mu^+}{\rho^-} = -\frac{q^2 \cos \phi^+}{\rho^-} \qquad \text{(positive, since } \rho^- < 0\text{)}$$

$$\frac{1}{f_2} = +\frac{g\mu^-}{\rho^+} = \frac{q^2 \cos \phi^-}{\rho^+} \qquad \text{(also positive, since } \rho^+ > 0\text{)}$$

and

$$a = \frac{\rho^+\chi^- - \rho^-\chi^+}{q^2 \cos \bar{\phi}} \qquad \text{(also, in general, positive)}$$

so that: $$\frac{\partial\gamma^+}{\partial\phi^-} = -\frac{1}{F} = -\frac{1}{f_1} - \frac{1}{f_2} + \frac{a}{f_1 f_2}$$

seems to represent a system which can have a negative value under certain conditions. However, there can never be a pronounced focus because the drift space is of the order of magnitude q^{-1} and the focal lengths are of the order of q, so these are very weak lenses in the case of a proton accelerator.

The matrix can, under certain circumstances, represent a slightly phase unstable condition. An example is the case in which $\phi^+ = \phi^- = -\bar{\phi}$, a condition which can only be met if $\bar{\phi} < 0$ and the particle spends about as much time in the phase defocusing region as in the phase focusing region of the tank (see Fig. 3). In this case, the drift space d between the two lenses becomes equal to zero and the lenses are of equal and opposed strengths. The "distance" $(\partial\phi^+/\partial\gamma^-)$ from input to output, as calculated from the $\left[M^+M^-\right]$ matrix, is then:

$$\left(\frac{\partial\phi^+}{\partial\gamma^-}\right) = \left(\frac{\rho^+ - \rho^-}{q^2 \cos \bar{\phi}}\right) .$$

However,

$$\rho^+ - \rho^- \cong \frac{\Delta(\gamma\beta)}{\bar{\gamma}\ \bar{\beta}} \cong \frac{\Delta\gamma}{(\bar{\gamma}\ \bar{\beta})^2} = \frac{1}{(\bar{\gamma}\ \bar{\beta})^2}\ \frac{eEL}{mc^2}\left\langle \cos\phi \right\rangle$$

and

$$\left(\frac{\partial\phi^+}{\partial\gamma^-}\right) \cong \frac{\dfrac{eEL}{mc^2(\bar{\gamma}\ \bar{\beta})^2}\left\langle \cos\phi \right\rangle}{\dfrac{eE\lambda}{2\pi mc^2}\ \bar{\gamma}\ \bar{\beta}\ \ \cos\bar{\phi}} \cong \frac{2\pi L}{\lambda(\bar{\gamma}\ \bar{\beta})^3}\ \frac{\left\langle \cos\phi \right\rangle}{\cos\bar{\phi}} \cong \frac{2\pi L}{\lambda(\bar{\gamma\beta})^3}$$

and one can write

$$\begin{bmatrix} \delta\phi^+ \\ \delta\gamma^+ \end{bmatrix} = \begin{bmatrix} 1 & + \dfrac{2\pi L}{\lambda(\bar{\gamma\beta})^3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta\phi^- \\ \delta\gamma^- \end{bmatrix}$$

in exact analogy to a drift space*.

Although as yet no detailed analysis has been done of the stepped β accelerator, for first order phase group stability it seems likely that some region near the "drift" condition just discussed ($\phi^+ = \phi^- = -\bar{\phi}$), which is in fact a defocusing condition, will be the border of the phase stability region. If phase compression is desired an average orbit should be chosen which makes only a small excursion into the phase defocusing region. There will be, as usual, some loss of accelerating efficiency by going to values of $\bar{\phi}$ much greater than -5° to -10°.

*This was pointed out by R.L. Gluckstern

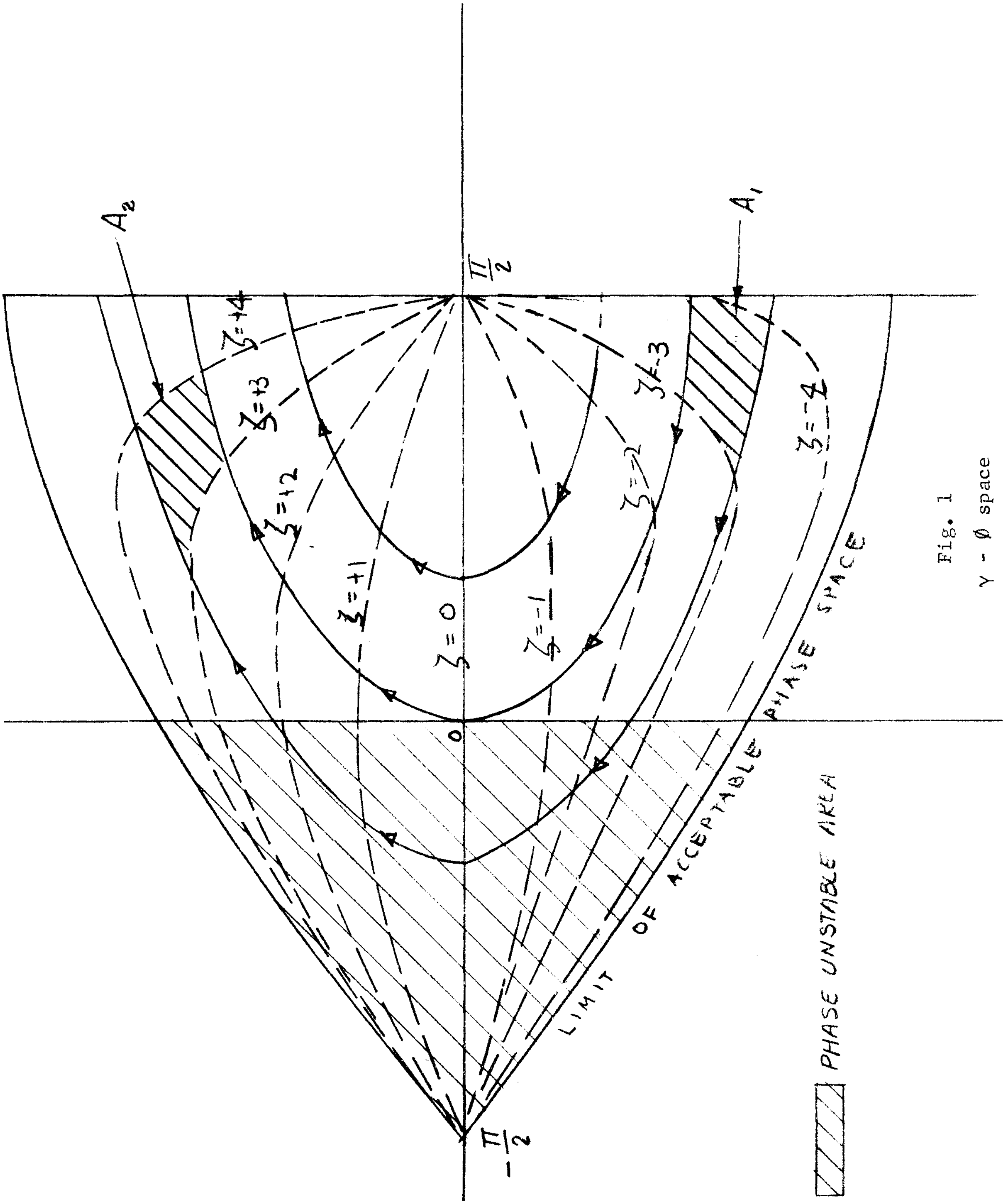
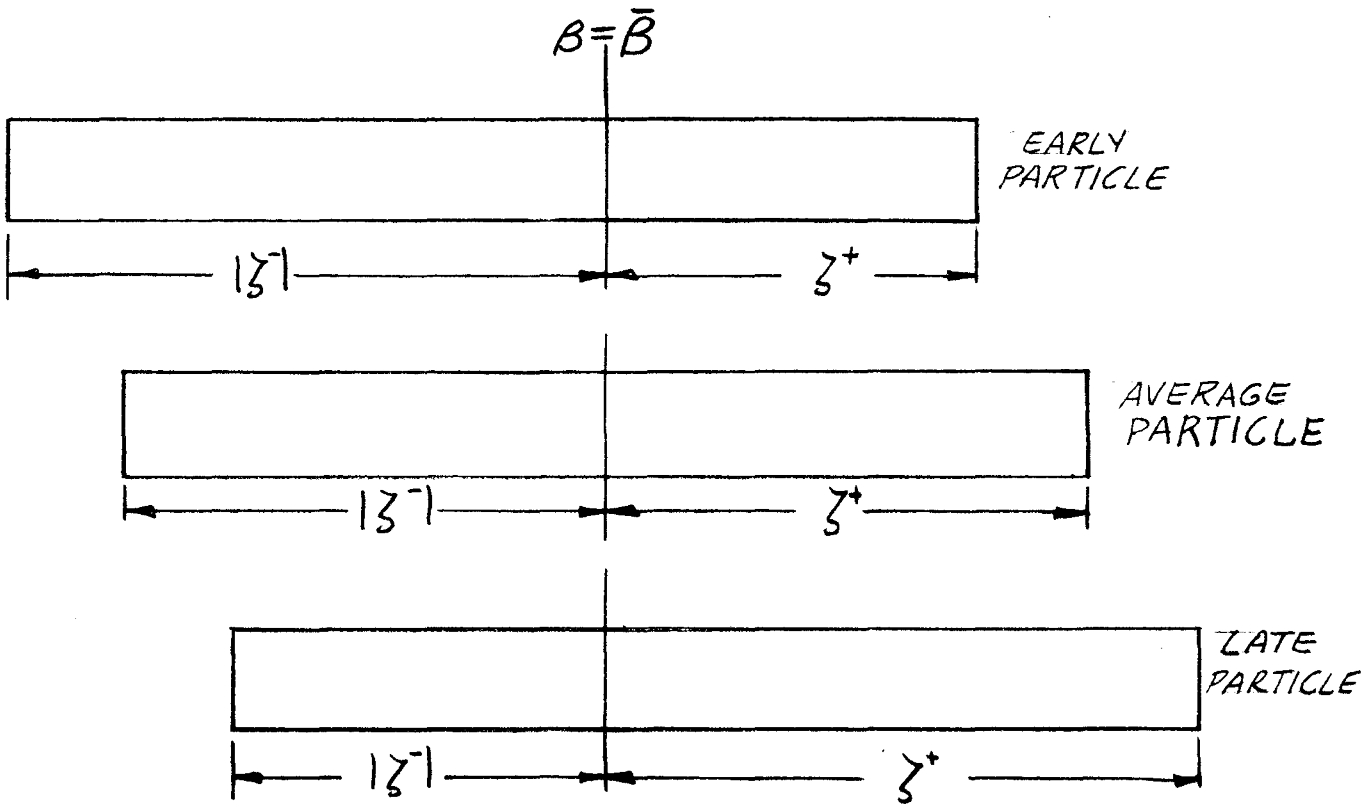

A2
A1
π/2
-π/2
0
ζ=+4
ζ=+3
ζ=+2
ζ=+1
ζ=0
ζ=-1
ζ=-2
ζ=-3
ζ=-4
LIMIT OF ACCEPTABLE PHASE SPACE
PHASE UNSTABLE AREA

Fig. 1
γ - ∅ space



$\beta = \bar{\beta}$
EARLY PARTICLE
$|\zeta^-|$
ζ^+
AVERAGE PARTICLE
$|\zeta^-|$
ζ^+
LATE PARTICLE
$|\zeta^-|$
ζ^+

Fig. 2

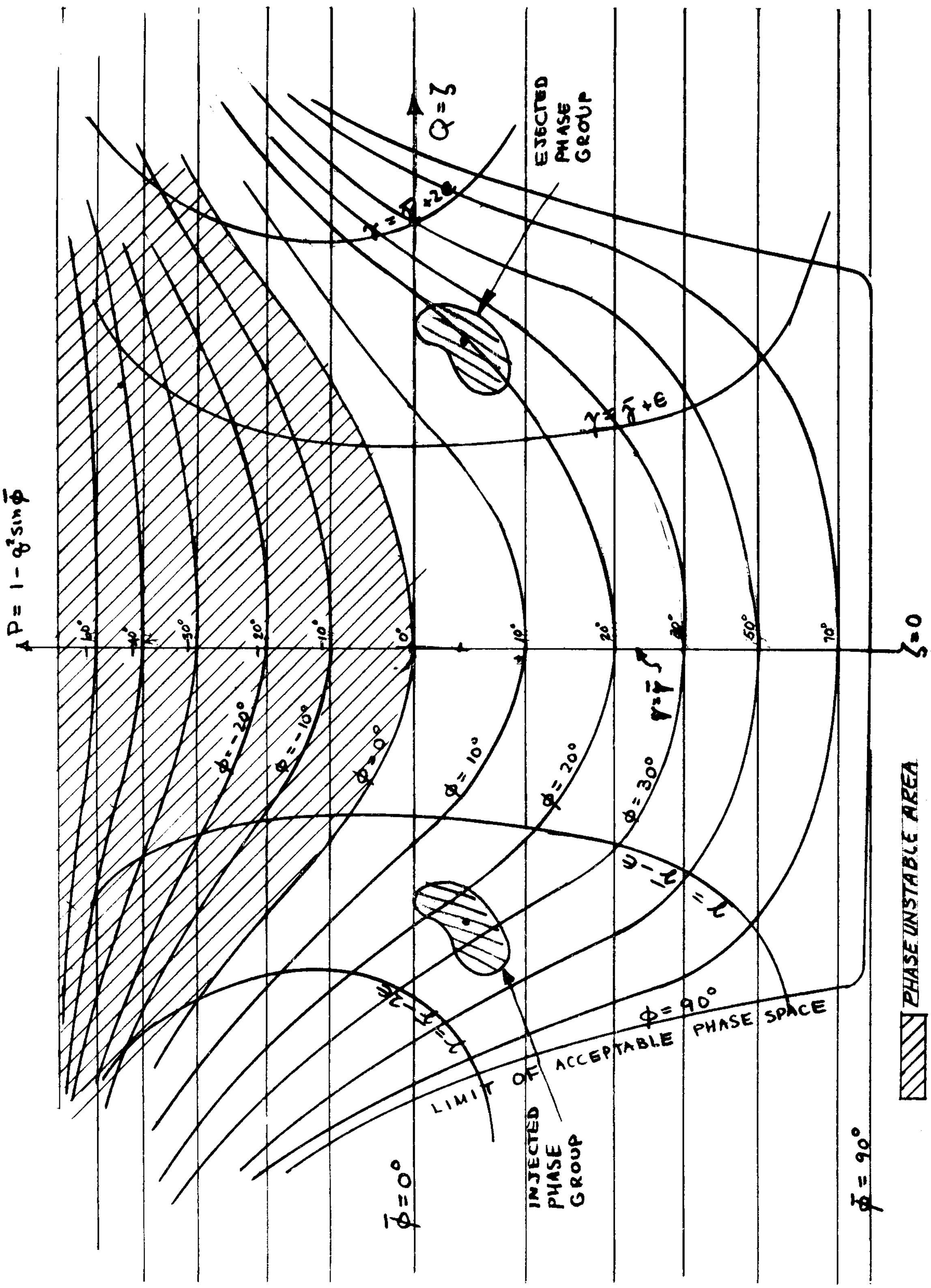


Fig. 3

P - Q space

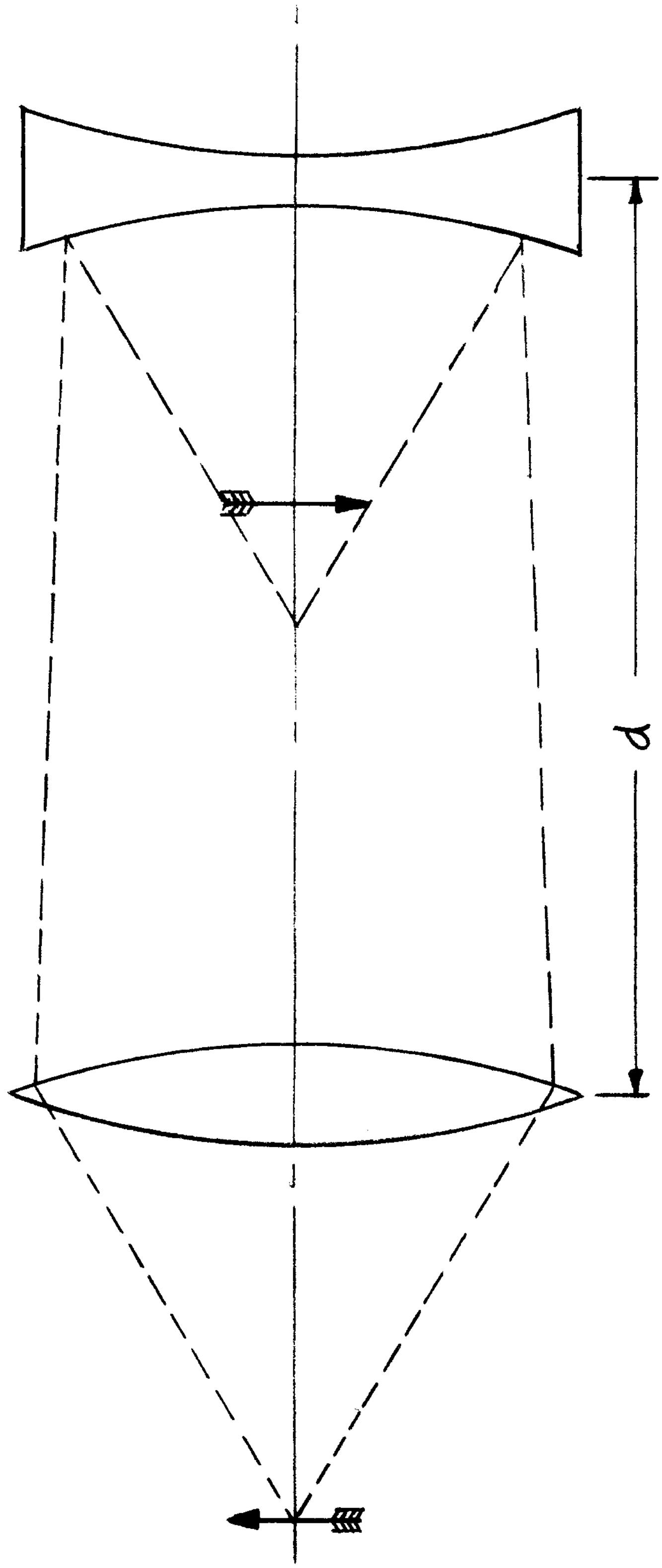


Fig. 4