STEPPED PHASE VELOCITY LINACS

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After some preliminary estimates had been made before of the parameters for a high energy stepped phase velocity linac, it was decided to proceed with a more detailed analysis. This was done analytically. Parallel to this, the computer approach was used; this part was done by H.S. Snyder. There was an interruption in this work because of his death. The numbers available at present apply, therefore, to a very specific case only. This is for an energy of 200 Mev, a field gradient of 8 Mv/m and a section length of 12 wavelengths, which at 1200 Mc/s is about 3 meters. These parameters, even at 200 Mev, are similar to those for a normal tapered phase velocity type linac. The problem here, however, is that one cannot speak of a synchronous phase. This makes it difficult to decide on entrance conditions because of the wide choice of parameters available. The first point to be considered, therefore, is how to properly choose a "reference" particle, in order to maximize the phase stable area about this "reference" particle. To study this, the γ-φ phase diagram is used. This is given in Fig. 1, drawn only to the right of the midpoint and showing the usual "fish" diagram. The trajectories near $\varphi = -\pi/2$ are close to what really is a phase stable point, but this is of no interest to the present case.

From the Hamiltonian approach ellipses are obtained, as shown. By taking a second integration, involving an elliptic integral, lines of constant lengths in λ are obtained, also indicated in Fig. 1, which are of particular interest. Two curves, as shown, refer to the "symmetric" case, such that on each side it marks half of an accelerator section length.

A particle entering the accelerator at some point on a line of constant length with phase φ and $\Delta\gamma = \gamma - \bar{\gamma}$, where $\bar{\gamma}$ is the γ corresponding to the constant phase velocity, will travel through the accelerator along the appropriate trajectory and wind up with the same φ and with a $\Delta\gamma$ of the same magnitude but with opposite sign. This symmetric case is useful because it is desirable to evaluate first of all how to let a particle go through in order to get maximum energy gain. In this particular case entering the first section at 200 Mev, one can choose any value for the wave velocity and also choose the entry phase of this first section. Therefore, the question is where the particle should be entered in the γ-φ diagram in order to obtain both good energy gain and phase stability. For the symmetric case this is evident; starting somewhere along the line, corresponding to $-z/2$, and ending up on the line, corresponding to $+z/2$, the maximum energy gain is obtained where the $\pm z/2$ curves have a maximum and minimum, respectively. This is indicated in Fig. 1 for the case of $z=12\lambda$. There is some reason to believe that this is the maximum anywhere for a given z and $\bar{\beta}$, including all classes of asymmetry, but this has not been proved. A curve indicating $\Delta\gamma$ as a function of starting phase for the symmetric class is given in Fig. 2. The energy gain cannot exceed the value $eEz/(Mc^2)$, also shown. For the case at hand, this value would correspond to 24 Mev and the indicated curve for $\Delta\gamma$ has its maximum value corresponding to 23 Mev at a starting point in phase of -25°. This starting point seems favorable because it results in nearly maximum obtainable energy gain. Proceeding further and including now not only the symmetric class but also nearby starting conditions leads to "islands", as shown in Fig. 1.

These refer to contours, around the maximum energy gain point, of starting points corresponding to constant energy gain.

Assuming now that from the foregoing results a "reference particle" is determined, then it is practical to consider the question of phase stability by examining the motion of particles with starting conditions close to that of the "reference particle". To simplify the problem, the following applies. A number of accelerator sectors have repetitive characteristics; the α, β and μ, used in the linear focusing matrices have close to the usual significance; the focusing strength in the linear approximation is maximized by some criterion and the reference ray (particle) is chosen on this basis.

It was tried to "guess" the limits of phase stability, also including non-linear effects. Part of this "guess" is to avoid reference rays which might run close to the boundary of the usual "fish" diagram and stay as close to its center as is possible, consistent with a decent energy gain. It was noted again that the symmetrical case is favorable as far as calculations are concerned because it is possible, for example, to write the expression for $\sin^2\mu/2$ for the case of reference rays starting on symmetric lines, as follows:

$$\sin^2\mu/2 = \frac{\frac{d}{d\varphi}\left[\Delta\gamma(\varphi)\right]^2}{a^2\cos\varphi_i + \frac{d}{d\varphi}\left[\Delta\gamma(\varphi)\right]^2}$$

where φ_i = initial phase and $a^2 = \frac{eE\lambda}{\pi mc^2}\,\bar{\gamma}^3\,\bar{\beta}^3$.

Writing now the usual transfer matrix for one section, without drift and for the case of high energy where the phase change is small, one gets:

$$\begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\left(\frac{1+\alpha^2}{\beta}\right)\sin\mu & \cos\mu - \alpha\sin\mu \end{bmatrix}$$

Numerically, it can be shown that

$$\frac{d}{d\varphi}\left[\Delta\gamma(\varphi)\right]^2 \quad \text{is small compared to } a^2\cos\varphi_i .$$

Therefore, rewriting $\sin\mu/2$, in the limit of small μ, the following expression is obtained:

$$\omega^2 \cong \frac{2\pi eE}{\lambda\ mc^2}\ \frac{1}{\bar{\gamma}^3\ \bar{\beta}^3}\ \sin\varphi .$$

This expression is identical to that for the tapered phase velocity linac. This is with the usual frequency for phase oscillations in the case where the phase slip is not very great. At lower energies the above has to be modified by the inclusion of the derivative term in the expression for $\sin^2\mu/2$.

It will be shown now that the machine behaves like a tapered phase velocity linac in the sense that the point of maximum energy gain, where the slope is zero (Fig. 2), also gives the point in which the phase focusing is neutral. Operating mostly on the defocusing side of the wave, the phase motion is unstable, because $\sin^2\mu/2$ is negative, but running mostly on the focusing side this is stable. The value of $\sin^2\mu/2$ is also given in Fig. 2. It varies from 1 through 0 to negative values. There is a check point available in the fact that at the value $-\pi/2$, the machine behaves like a tapered phase velocity linac, oscillations with respect to the reference ray are just phase oscillations in a machine with zero energy gain. Calculating the frequency for these phase oscillations result in an independent value for this point.

The expression $\sin^2\mu/2$ is a function of starting phase, as shown, being zero where $\Delta\gamma$ is maximum. Its behavior strongly resembles a sine curve displaced by 90°. In a tapered phase velocity linac $\Delta\gamma$ follows a cosine curve and $\sin^2\mu/2$ a sine curve. It was found that the line of phase neutrality could be obtained analytically from the general trajectories. This was confirmed with the computer program. This phase neutral line is indicated in Fig. 1, it connects starting conditions with the characteristic that $\sin^2\mu/2$ in the matrix is zero. This line separates a region of phase instability from a region of phase stability, as indicated.

The next step was to evaluate the matrix quantity β which would approach β_{max} in practical situations. This was done both analytically and with the computer. The results are also given in Fig. 2, where $1/\beta$ is plotted. The value of β becomes infinite where $\Delta\gamma$ is maximum and where phase stability is lost. For negative values of $\sin^2\mu/2$, the behavior of $\sin^2\mu/2$ and $1/\beta$ as a function of φ_i is similar, as indicated by the single line in Fig. 2. Apparently, on a linear basis, β and μ are equivalent as a measure of how stable a particle is for small displacements from the reference ray.

For the symmetric case favorable results are now obtained by going to the point of maximum energy gain and then along a line of constant distance corresponding to -6λ and finally to take the appropriate starting conditions in the stable region. Doing this, it was guessed that the focusing strength gained would have the same relation to energy gain lost as in the tapered phase velocity linac; probably a 10% to 15% loss of energy gain. It was felt by H.S. Snyder that this was too complicated because it would necessitate investigating each section and computing constant length curves to specify starting conditions. A prescription was found for the computer program whereby the point of maximum energy gain is located and then, instead of going along a curve of constant length to move sideways (see Fig. 1), keeping $\Delta\gamma$ constant, and to obtain a starting condition for which the energy gain had dropped by a certain amount. No results have been obtained yet with this program for the reason stated before.

Discussion

R. L. Gluckstern (Yale): Do you use a non-relativistic approximation to get the γ-φ phase diagram?

L. Smith (LRL): No. This is an approximation to $H = \bar{\gamma}\gamma - \overline{\gamma\beta}\,\gamma\beta -$ etc., and taking only first order terms in $\Delta\gamma$. The linear $\Delta\gamma$ drops out, and we get $[\Delta\gamma]^2$ plus small terms.

R. L. Gluckstern (Yale): In the γ-φ phase diagram, is the horizontal line at $\beta = \bar{\beta}$ therefore an axis of symmetry?

L. Smith (LRL): Yes. If we count only the first term, then it is a completely symmetric diagram and this is the familiar approximation for this type of work. From the computer results, the phase diagram really is asymmetric, so things came out a little lopsided when trying to make a comparison, but indeed it was a very small difference, and it really does not matter except when we were worrying about the program being correct. So the usual "fish" is in this case certainly an excellent approximation.

R. L. Gluckstern (Yale): Do you take a fixed point in the phase diagram and find out what the focusing matrix is?

L. Smith (LRL): I take reference rays in the symmetric class and figure out the differential stability with respect to those reference rays.

R. L. Gluckstern (Yale): If you rewrote the $\sin^2\mu/2$ expression in terms of the average phase, might not the second term in the denominator be comparable to the first? It seems that it is really the average φ you want there?

L. Smith (LRL): Yes. I think for all practical purposes, for the kind of parameters we are talking about, that it might pay to rewrite these in terms of effective phases. At present, we are still interested in the general problem.

R. B. R-Shersby-Harvie (CERN): Does phase stability correspond very closely to gain?

L. Smith (LRL): For the symmetric type of orbit it does; there is an excellent energy gain at the point where $\sin^2\mu/2$ is zero. It is not so obvious what the relationships are for the nonsymmetric situation.

R. L. Gluckstern (Yale): What fraction of an oscillation is one section?

L. Smith (LRL): Note that the line marked 8 in Fig. 1 is one quarter of an oscillation for small oscillations about the phase stable point. This line is just about 8 wavelengths, and beyond that, it curves around. So 32 wavelengths, about six sections, would be one oscillation.

E. D. Courant (BNL): Are the computer programs in better shape now?

L. Smith (LRL): No. They are not; that is one reason that we have not had any further results yet.

R. B. Neal (Stanford): If you do extend this analysis to the beam loading case, you will probably find that your injection phase will want to shift around such that it compensates for the energy subtraction from the cavity. The general trend will be the same, at least in the electron case. We found that the net effect on the phase from the induced wave from the electron beam was such that the maximum energy gain condition corresponds to a zero net effect on the traveling wave passing through the accelerator structure. In the first part of the structure, you are causing phase shift in one direction and in the last part less phase shift in the other direction. These two effects exactly cancel for the proper phase angle at injection to give you the maximum energy gain.

J. P. Blewett (BNL): Do you feel now that if it turned out that you do not get less expensive construction by this method that you still would be inclined to use it?

L. Smith (LRL): I doubt it. However, this type of accelerator looks feasible. It represents a degree of freedom in the design which is not cut off because of beam dynamics.

L. C. Teng (Argonne): Do you have any idea what the size is of the stable phase space? Do you expect it to differ much from the usual case?

L. Smith (LRL): I do not expect it to be very different in practice. The total phase area shown in Fig. 1 is much larger than the ones we talk about in the corresponding tapered phase velocity case. So the equivalent areas would be about the same if we consider that the corresponding gradient for the tapered phase velocity case were to be tapered in synchronous phase. The phase acceptance area might be something of the same general size. Because we are drawing the stepped phase velocity case for this whole range of phases and the same gradient, the picture is much bigger because of the $-\pi/2$ to $+\pi/2$ range.

L. C. Teng (Argonne): I have a feeling that the linear theory only holds for a small region.

L. Smith (LRL): We were only thinking of criteria to lay out a design in which we would really run trajectories and see what happens. The feeling was that if we were to make the linear focusing as strong as possible, that this would be the best for the nonlinear case too.

K. Johnsen (CERN): There is an approach which possibly could be considered. Suppose you put the loops on top of each other so that you have in effect, a normal synchronous phase angle plus variations, a perturbation. Then you can use the normal tapered phase velocity system to see just where the stopbands would be.

L. Smith (LRL): A procedure such as you suggest would probably be quite interesting. It certainly would be a very good thing to try. At the present stage we are, however, after the principle of operation and after the prescription for computer work.

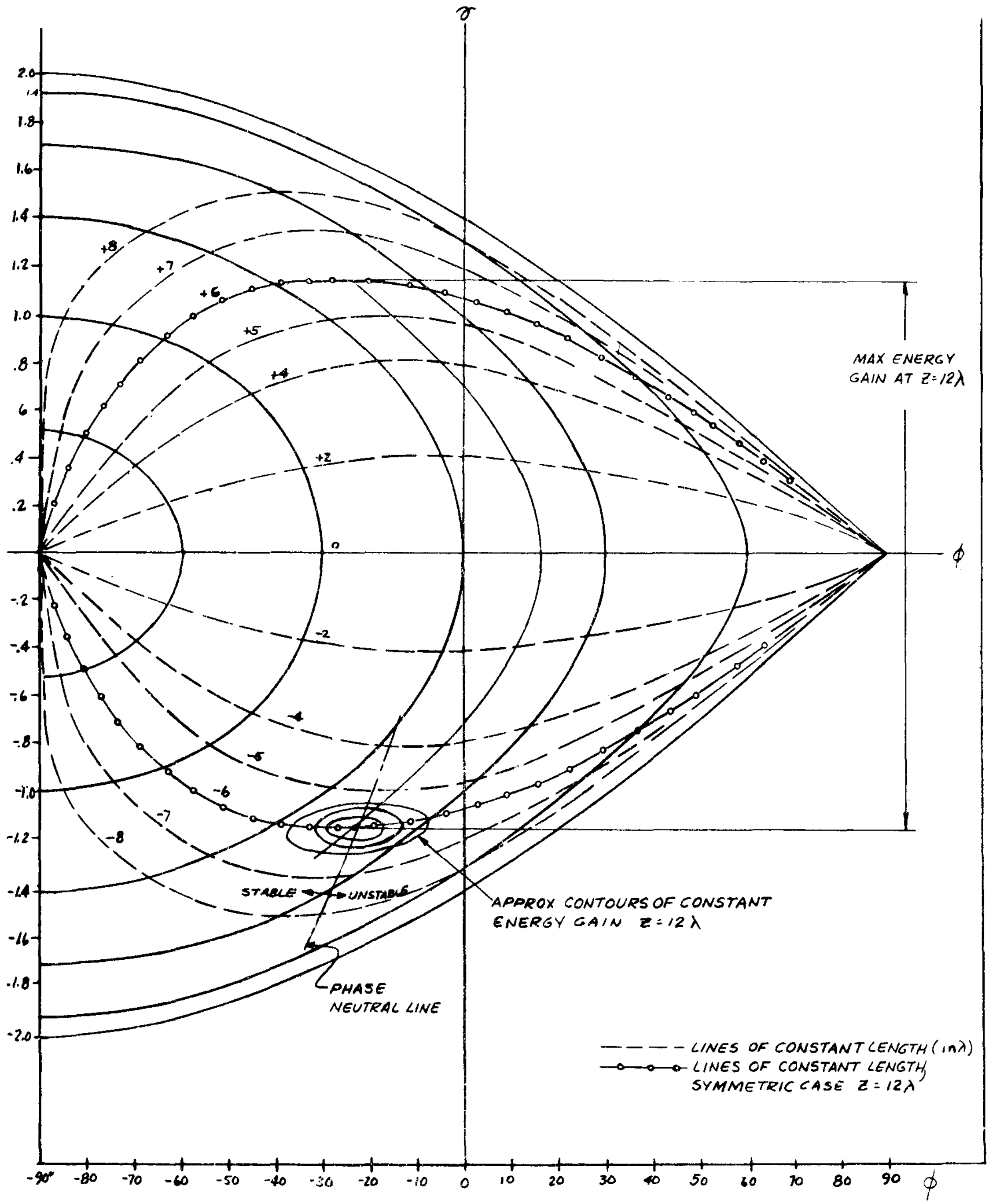

FIGURE 1
PHASE DIAGRAM FOR 200 MEV,
1200 MC, CONSTANT PHASE VELOCITY

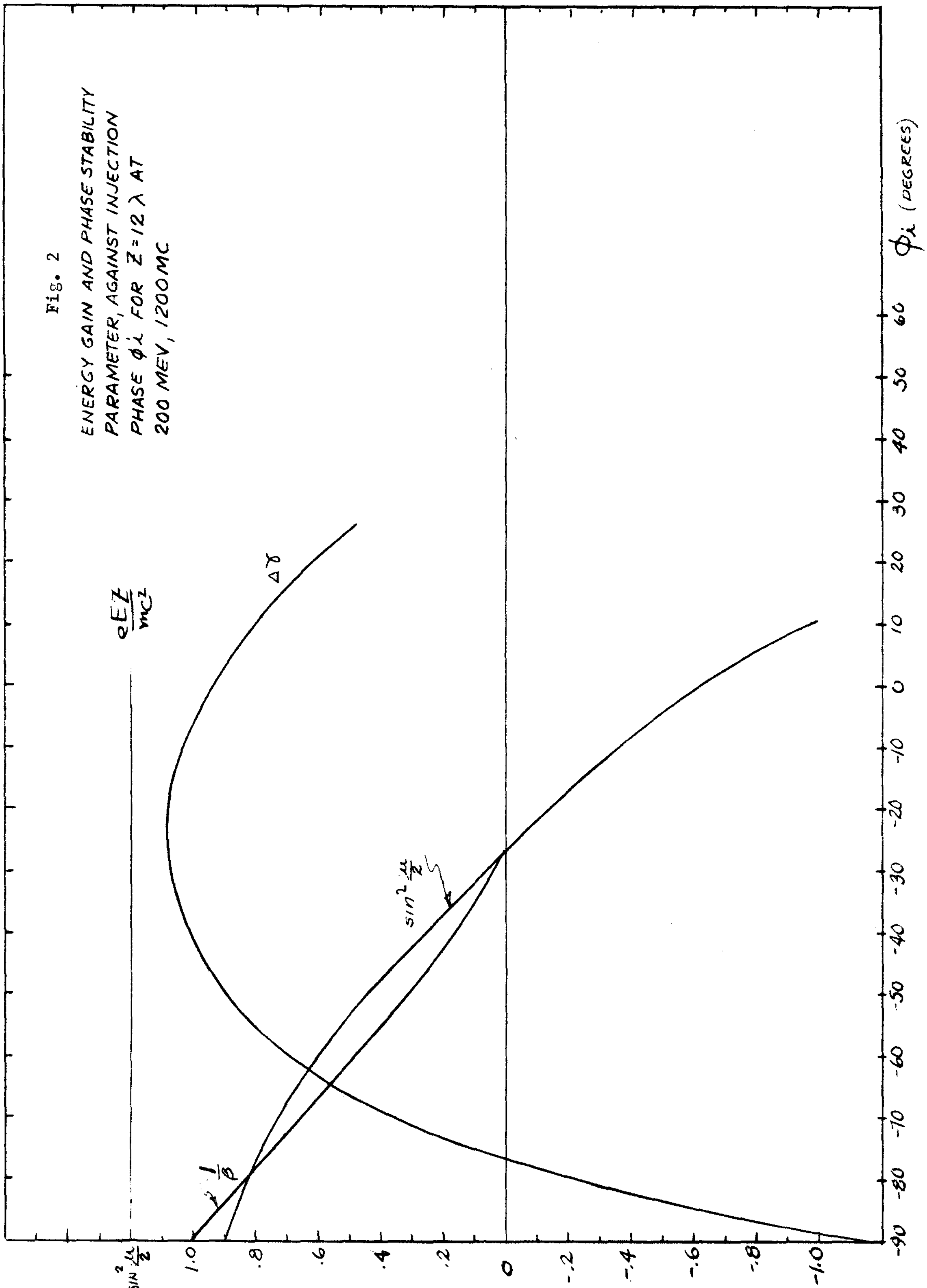

Fig. 2

ENERGY GAIN AND PHASE STABILITY PARAMETER, AGAINST INJECTION PHASE ϕ_i FOR $Z = 12\lambda$ AT 200 MEV, 1200 MC