

October 22, 1963

STEADY STATE BEHAVIOR OF A RING OR OF A CHAIN  
OF COUPLED CIRCUITS

D. E. Nagle and E. A. Knapp  
Los Alamos Scientific Laboratory

We are going to treat briefly the characteristics of a linear accelerator cavity, using an equivalent lumped circuit, which will be familiar to most of you. We have found this useful at Los Alamos for showing the effects of coupling and of perturbations in each cavity, and for obtaining numerical results on the system.

Consider a ring of N coupled circuits as shown in Fig. 1, where k is the dimensionless coupling coefficient between two circuits. The equation of the n<sup>th</sup> loop is

$$i_n (2j\omega L + R + \frac{1}{j\omega C}) + k j\omega L (i_{n-1} + i_{n+1}) = 0 \quad (1)$$

$$i_n (1 + \frac{\omega_o}{\omega} \cdot \frac{1}{jQ} - \frac{\omega_o^2}{\omega^2}) + \frac{k}{2} (i_{n-1} + i_{n+1}) = 0 \quad (2)$$

$$i_n (1 - \frac{\omega_o^2}{\omega^2}) + \frac{k}{2} (i_{n-1} + i_{n+1}) = 0 \quad (3)$$

(the losses are lumped into  $\omega_o$ )

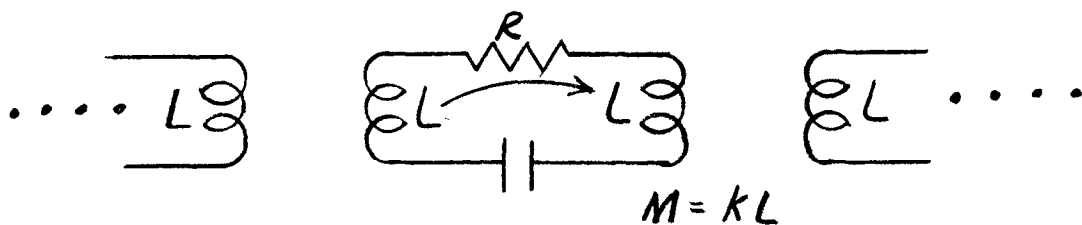


FIG 1

$$\frac{\omega_o^2}{\omega^2} = \frac{\omega_o^2}{\omega^2} - \frac{\omega_o}{\omega} \frac{1}{jQ} \quad (4)$$

Notice that

$$D^2 i_n = i_{n+1} + i_{n-1} - 2i_n$$

So

$$\frac{k}{2} D^2 i_m + i_m \left( \frac{k}{2} \cdot 2 + 1 - \frac{\omega_o^2}{\omega^2} \right) = 0$$

With the time dependence put back in, this is the wave equation in finite difference form.

Solutions are of the form

$$e^{i \cdot 2\pi q n / N} = i_n(q) \quad \text{provided that} \quad (5)$$

$$\left( 1 - \frac{\omega_o^2}{\omega^2} \right) + k \cos \frac{2\pi q}{N} = 0 \quad (6)$$

if  $q$  is integral (5) satisfies the boundary condition for a ring:

$$i_{N+1} = i_1 \quad (7)$$

Here,  $q$  is called the mode number: so  $i_n(q)$  is the current in the  $n^{\text{th}}$  circuit for the  $q^{\text{th}}$  mode if there

are  $N$  circuits connected in a ring. Since there are  $N$  degrees of freedom to the system, the set  $i_n(q) = e^{2\pi i q n / N}$ ,  $q = 1 \dots N$  form a complete set. The set  $i_n^*(q)$  are not independent, since

$$i_n(q) = i_{N-n}^*$$

The solutions  $i_n(q)$  and  $i_n(q)^*$  represent waves running around the ring in opposite directions. Note that from (6), they have the same frequency.

We can make up standing wave solutions as well:

$$u_n(q) = \cos 2\pi q n / N = \frac{i_n(q) + i_n(q)^*}{2i} \quad (9)$$

$$v_n(q) = \sin 2\pi q n / N = \frac{i_n(q) - i_n(q)^*}{2i} \quad (10)$$

Again, either the  $u$ 's or the  $v$ 's can form a basis.  $i_n(q)$  and  $i_n(q)^*$  belong to the frequency given by Eq. (6), and so must  $u_n(q)$  and  $v_n(q)$  because of the linearity of (3). From (9)

$$u_{-1}(q) = u_1(q) \quad \text{and (3) for } m = 0 \quad (11)$$

becomes

$$u_0 \left( 1 - \frac{\omega_0^2}{\omega^2} \right) + \frac{k}{2} (u_{-1} + u_1) = 0$$

or

$$u_0 \left( 1 - \frac{\omega_0^2}{\omega^2} \right) + k u_1 = 0 \quad (12)$$

Similarly,

$$u_N \left( 1 - \frac{\omega_o^2}{\omega^2} \right) + k u_{N-1} = 0 \quad . \quad (13)$$

Consider a linear chain of circuits. We collect the equations:

$$i_n \left( 1 - \frac{\omega_o^2}{\omega^2} \right) + \frac{k}{2} (i_{n-1} + i_{n+1}) = 0 \quad (3)$$

$$i_o \left( 1 - \frac{\omega_o^2}{\omega^2} \right) + k i_1 = 0 \quad (12)$$

$$i_N \left( 1 - \frac{\omega_o^2}{\omega^2} \right) + k i_{N-1} = 0 \quad (13)$$

These govern the currents in the linear chain with half circuits on each end; as in Fig. (2).

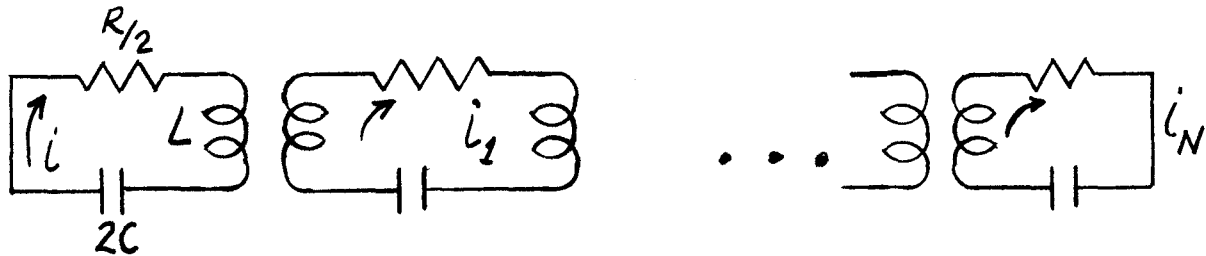


FIG. 2

The eigenfunctions are

$$i_n(q) = \cos \frac{\pi n q}{N} \quad (14)$$

with the eigenvalues

$$\frac{\omega^2}{\omega_0^2} = (1 + k \cos (\pi q/N))^{-1} \quad (15)$$

corresponding to (5) and (6), respectively. The factor of 2 appears because the wave travels down the chain and back before reinforcing the current in the 0<sup>th</sup> loop.

$$q = 0 \text{ is called the } 0 \text{ mode } \omega^2(q = 0) = \frac{\omega_0^2}{1+k}$$

this has the lowest fig. if  $k > 0$ .

$$q = N \text{ is called the } \pi \text{ mode } \omega^2(q = N) = \frac{\omega_0^2}{1-k}$$

this has the highest fig. of all the modes,

$k > 0$ .

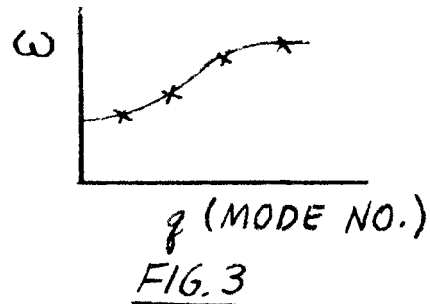
$q = \frac{N}{2}$ ,  $N$  even, is called the  $\pi/2$  mode

$$\omega^2(q = \frac{N}{2}) = \omega_0^2.$$

Notice that  $\omega(N - q) = \omega(q)$  from Eq. (6). The set of resonances  $\omega^2(q)$  for  $q = 0 \dots N$  are called the pass-band and  $[\omega(q = N) - \omega(q = 0)] = \text{band width}$ . The resonances are sprinkled onto the curve, which is given by

$$\omega^2 = \frac{\omega_0^2}{1 + k \cos (\pi q/N)} \quad (16)$$

as is shown in Fig. 3. The width of each resonance



is governed by its Q-value, namely

$$\frac{1}{Q_q} \approx \frac{1}{Q} \frac{\omega}{\omega_0} \quad (17)$$

which has the value for the  $\pi$ -mode

$$\frac{1}{Q_\pi} = \frac{1}{Q} \frac{1}{(1 - k)^{1/2}} \quad (18)$$

In the limit of very small losses, the eigenfunctions are  $\cos(\pi m q / N) = i_m^{(q)}$ . One practical problem is that of the spectrum generated by  $(N + 1)$  cavities coupled in a chain, where the lowest mode of a single cavity is isolated from higher modes of a single cavity, and for  $(N + 1)$  cavities we still have the simple spectrum described by (6). If we wish to run on the  $\pi$ -mode, the next mode is displaced ( $k \ll 1$ )

$$\frac{\Delta\omega}{\omega} = \frac{k\pi^2}{4N^2(1 - k)}$$

e.g.,  $k = 0.1N = 32 \frac{\Delta\omega}{\omega} \approx 3 \times 10^{-4}$  but this is of the order of  $1/Q$  for the L-band structures. Hence 32 cells may be satisfactory for  $k = 0.1$ . The parameter  $k$  may, in practical structures, be a strong function of  $\omega$ , which distorts the dispersion curve. It is desirable from the standpoint of  $\pi$ -mode operation to have  $k(\omega)$  large near the top of the passband.

Now, consider the energy and power relationships in the  $N$  coupled circuits needed. The circuit equations are

$$i_n Z_n + kj\omega L(i_{n-1} + i_{n+1}) = \epsilon_n \quad (19)$$

where  $n = 0 \dots N$ . From this it follows that

$$R\ell \sum_m^N i_n^* \epsilon_n = \sum_m^N i_n^* i_n R_n + R\ell \left[ kj\omega L i_m^* i_{m-1} \right] \quad (20)$$

Equation (20) is a statement regarding energy flow. The left hand sum is the power fed into the tank from drive loops from the  $m^{\text{th}}$  to the  $N^{\text{th}}$  cells, the right hand sum is the power dissipated in these cells, and the term on the right is the power transmitted from the  $m^{\text{th}}$  to the  $(m - 1)^{\text{th}}$  cell by the coupling mechanism. This requires a phase shift between  $i_m$  and  $i_{m-1}$  if power flows between these cells, since then

$$-W_m = - \sum_m^N i_n^* i_{n-1} R_{n-1} = R \ell \left[ k j \omega L i_m^* i_{m-1} \right] \quad (21)$$

Example:  $\pi$ -mode  $i_m \approx (-1)^m e^{i\varphi_m}$

$$-W_m \approx k \omega L (\varphi_m - \varphi_{m-1}) = (N - m)R + \frac{R}{2}$$

$$\varphi_m - \varphi_{m-1} = \frac{2 \sqrt{1 - k}}{k Q_0} \left( N - m + \frac{1}{2} \right) \quad (22)$$

( $\pi$ -mode)

Integrating with respect to  $m$  from 1 to  $N$

$$\Delta \varphi = \int_1^N \frac{d\varphi}{dm} dm = \frac{(1 - k)^{1/2}}{k Q_0} (N) (N - 1) \quad (23)$$

( $\pi$ -mode)

Comparing this with the mode spacing requirement

$$\frac{\Delta \varphi}{\omega} = \frac{k \pi^2}{4 N^2 (1 - k)} < \frac{1}{Q} = \frac{1}{Q_0 (1 - k)^{1/2}} \quad (24)$$

we see  $\Delta \varphi \lesssim 2$  radians.

The relationship between the two statements comes about because the mode spacing determines the group velocity and hence the rate of energy flow down the tank.



We now consider perturbations in the ring. In order to discuss, in the single mode or lumped-constant model, the effect of frequency errors in the individual cells, we imagine that, as before, the individual cells are described by resonant frequencies  $\omega_{on}$ . Small errors in the size of the cells cause the  $\omega_{on}$  to be slightly different from  $\omega_o$ , the average frequency of the cells. Express this as follows:

$$0 = i_n (2j\omega L_n + \frac{1}{j\omega C} + R_n) + jk\omega \left[ i_{n-1} (L_{n-1} L_n)^{1/2} + i_{n+1} (L_{n+1} L_n)^{1/2} \right] \quad (25)$$

$$\omega_n^2 = \frac{1}{2L_n C} \left( 1 + \frac{1}{jQ} \right) \equiv \omega_o^2 (1 + \epsilon_n) \quad (26)$$

$$\omega_o^2 = \left\langle \omega_n^2 \right\rangle_{AV} \quad (27)$$

The circuit equations can now be written as

$$(M + \delta M) (i + \delta i) = \left( \frac{1}{\omega} + \delta \frac{1}{\omega} \right) (i + \delta i) \quad (28)$$

Here  $i$  is the current vector ( $i_1 \dots i_N$ ),  $\delta i$  is the perturbed current vector.

$M$  is the matrix with elements

$$M_{mn} = \frac{1}{\omega} \left( \delta_{mn} + \frac{k}{2} \delta_{m+1,n} + \frac{k}{2} \delta_{m-1,n} \right) \quad (29)$$

$\delta M$  is the perturbed matrix and has the elements

$$(\delta M)_{mn} = -\frac{1}{\omega_0^2} \left[ \epsilon_n \left( \delta_{mn} + \frac{k}{4} \delta_{m+1,n} + \frac{k}{4} \delta_{m-1,n} \right) + \epsilon_{n-1} \frac{k}{4} \delta_{m+1,n} + \epsilon_{n+1} \frac{k}{4} \delta_{m-1,n} \right] \quad (30)$$

where  $\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$

The eigenfunctions of  $M$  are

$$i_\ell(q) = N^{-1/2} \exp(iq\ell \cdot 2\pi/N) \quad (31)$$

with corresponding eigenvalues

$$(\omega^{(\ell)})^{-2} = \omega_0^{-2} (1 + k \cdot \cos 2\pi\ell/N) \quad (32)$$

The usual perturbation theory prescriptions are, for first order:

$$\delta(\omega_\ell^{-2}) = i^{\ell*} \cdot (\delta M) \cdot i^\ell \quad (33)$$

$$\delta i^{(\ell)} = \sum_{q \neq \ell} \frac{[i^{*(q)} \cdot (\delta M) \cdot i^{(\ell)}] i(q)}{(\omega^{(\ell)})^{-2} - (\omega^{(q)})^{-2}} \quad (34)$$

Carrying out these rules, we obtain

$$\frac{\delta \omega_{\ell}^{-2}}{\omega_{\ell}^{-2}} = \frac{1}{N} \sum_n \epsilon_n = \langle \epsilon \rangle_{AV} \quad (35)$$

and

$$\delta_i^{(\ell)} = \sum_{r=1}^{N-1} i^{(\ell+r)} \tilde{\epsilon}_r \frac{1 + \frac{1}{2} k \left( \cos \frac{2\pi\ell}{N} + \cos \frac{2\pi}{N} (\ell + r) \right)}{k \left( \cos \frac{2\pi\ell}{N} - \cos \frac{2\pi}{N} (\ell + r) \right)} \quad (36)$$

where  $\tilde{\epsilon}_r$  is the Fourier transform of the errors:

$$\tilde{\epsilon}_r = -\frac{1}{N} \sum_{p=1}^N \epsilon_p \exp \left( \frac{2\pi i}{N} p r \right) \quad (37)$$

Equation (35) says that the fractional frequency shift of any mode is the same, and equal to the fractional frequency error averaged over the  $N$  cells. This result breaks down for very small  $k \sim \epsilon$  since the expansion parameter is basically  $(\epsilon/k)$ . Equation (36) states that the admixture of the  $(\ell + r)^{\text{th}}$  mode in the perturbed currents  $\delta_i^{(\ell)}$  is  $\tilde{\epsilon}_r$  times a coefficient  $a_r$ :

$$a_r \equiv \frac{1 + \frac{k}{2} \left( \cos \frac{2\pi\ell}{N} + \cos \frac{2\pi}{N} (\ell + r) \right)}{k \cos \left( \frac{2\pi\ell}{N} - \cos \frac{2\pi}{N} (\ell + r) \right)} \quad (38)$$

$$\text{Example: } \begin{cases} 0 \text{ - mode, } \ell = N \\ k = 1 \end{cases}$$

$$a_r = \frac{1 + \frac{1}{2} (1 + \cos \frac{2\pi r}{N})}{1 - \cos \frac{2\pi r}{N}}$$

for  $r \ll N$  we have

$$a_r \approx \frac{1 + \frac{1}{2} (1 + 1)}{\frac{1}{2} \left(\frac{2\pi r}{N}\right)^2} = \frac{8N^2}{4\pi^2 r^2}$$

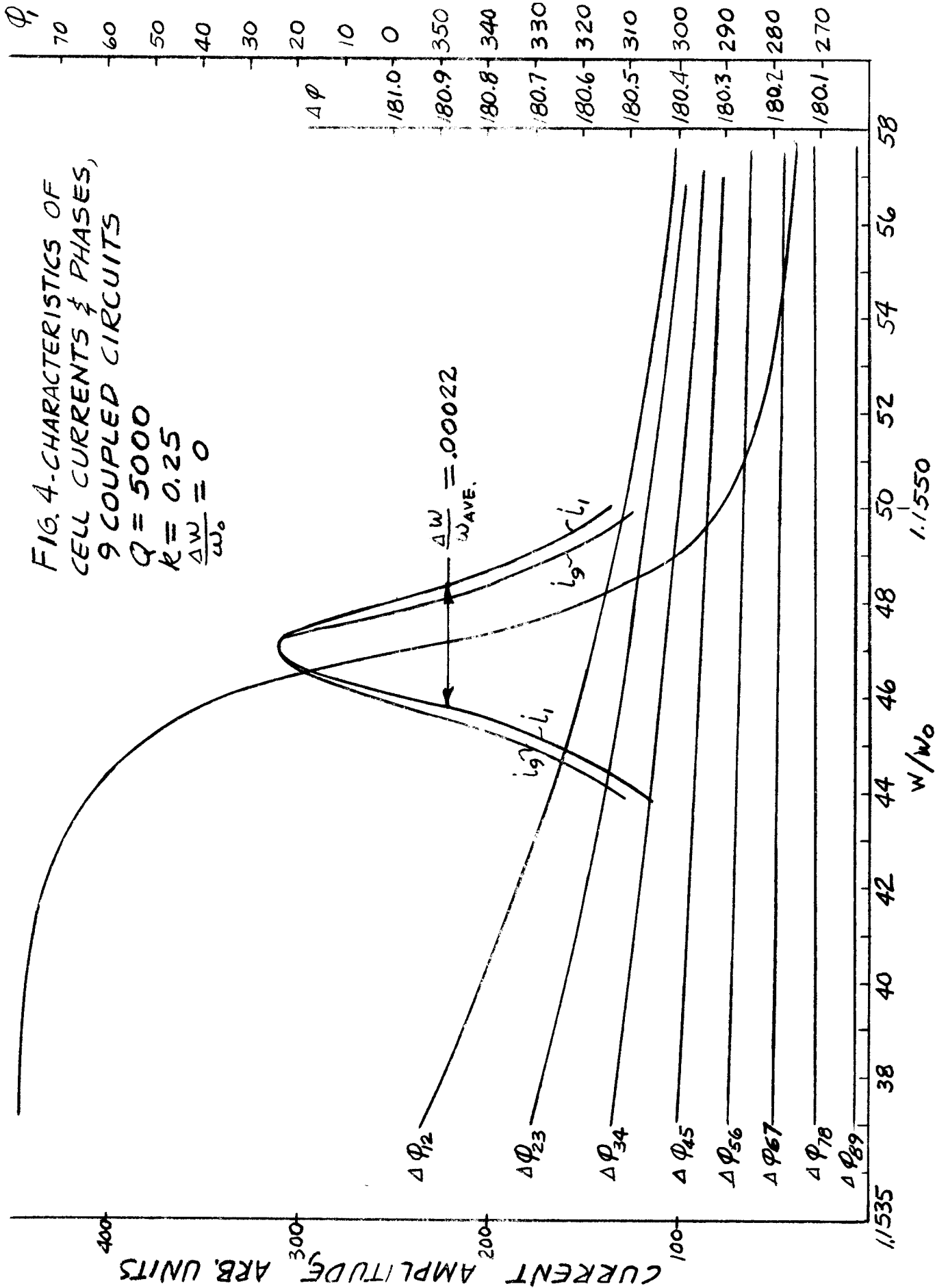
The result for this example is essentially that of Alvarez.<sup>(1)</sup> However, Eq. (36) holds for any coupling strength and any mode.

For  $k \ll 1$ , Eq. (38) reduces to

$$a_r = \left[ k \cos \left( \frac{2\pi \ell}{N} - \cos \frac{2\pi(\ell + r)}{N} \right) \right]^{-1}$$

This dependence on  $1/k$  is exhibited in Knapp's calculation. It points out the fact that the tank flattening problem depends on  $k^{-1}$ .

As an experimental verification, we took a circuit of  $N$  lumped cells ( $N$  variable) with a current generator at one end and a matching half cell at the other. Figure 4 shows a system in which  $N = 9$ ,  $Q = 5000$ ,



$k = 0.25$ , (a bandwidth of 25%) and the frequencies of all cells are taken to be equal. These are in the  $\pi$ -mode. Cell #1 is farthest from the current source and Cell #9 is closest to the generator.

For this case, it can be seen that the phase shift between cavities farthest from the generator ( $Q_{12}$ ) is not exactly  $\pi$ , but misses it by  $0.6^\circ$  at resonance. This particular case transmits power by having the amplitude in all nine cells exactly the same (at  $\omega/\omega_0 = 1.1547$ ) and the phase shift between cells  $\Delta\phi$  slightly different from  $\pi$ . You can see that  $\Delta\phi$  becomes very small as you get closer to the generator.

In the next case, as shown in Fig. 5, we introduce random errors in tuning. The  $\Delta\phi$ 's stay virtually the same, and the effect of the errors is to change the voltage from cavity to cavity and thus the currents are different. The errors are randomly distributed in a linear fashion, and of order  $10^{-3}$  to  $10^{-4}$ .

Figure 6 shows a case of  $N = 18$  cells, in which the maximum of each set of random errors is indicated on each curve. This shows a strong correlation. The current varies smoothly from cell to cell even though the individual errors are random. An error in one cell influences the currents throughout the cavity. The error of 0.5% is rather large, so is 0.05%, for that matter.

When tuning and flattening a circuit like this, one must go through the arithmetic in order to decide

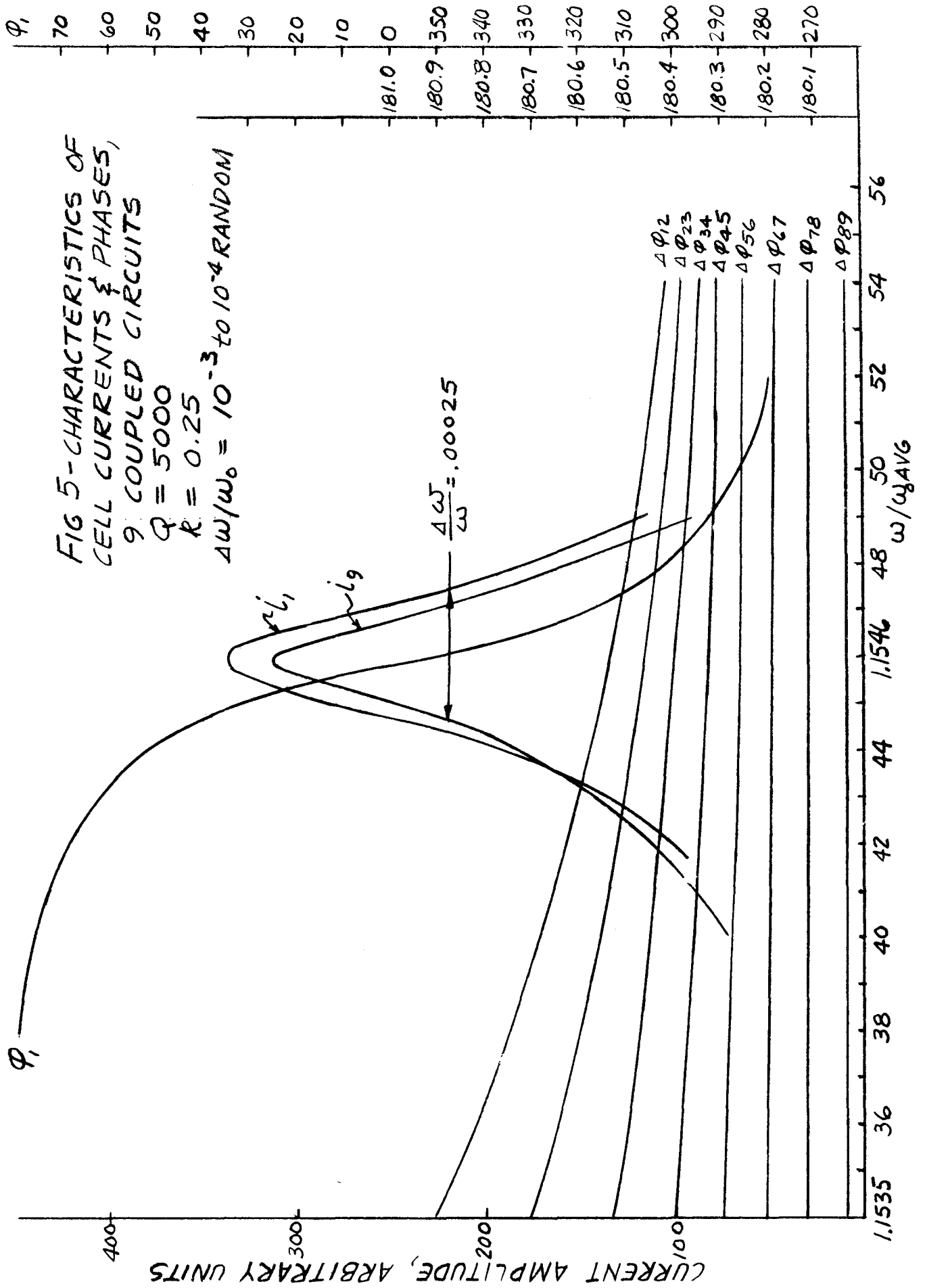
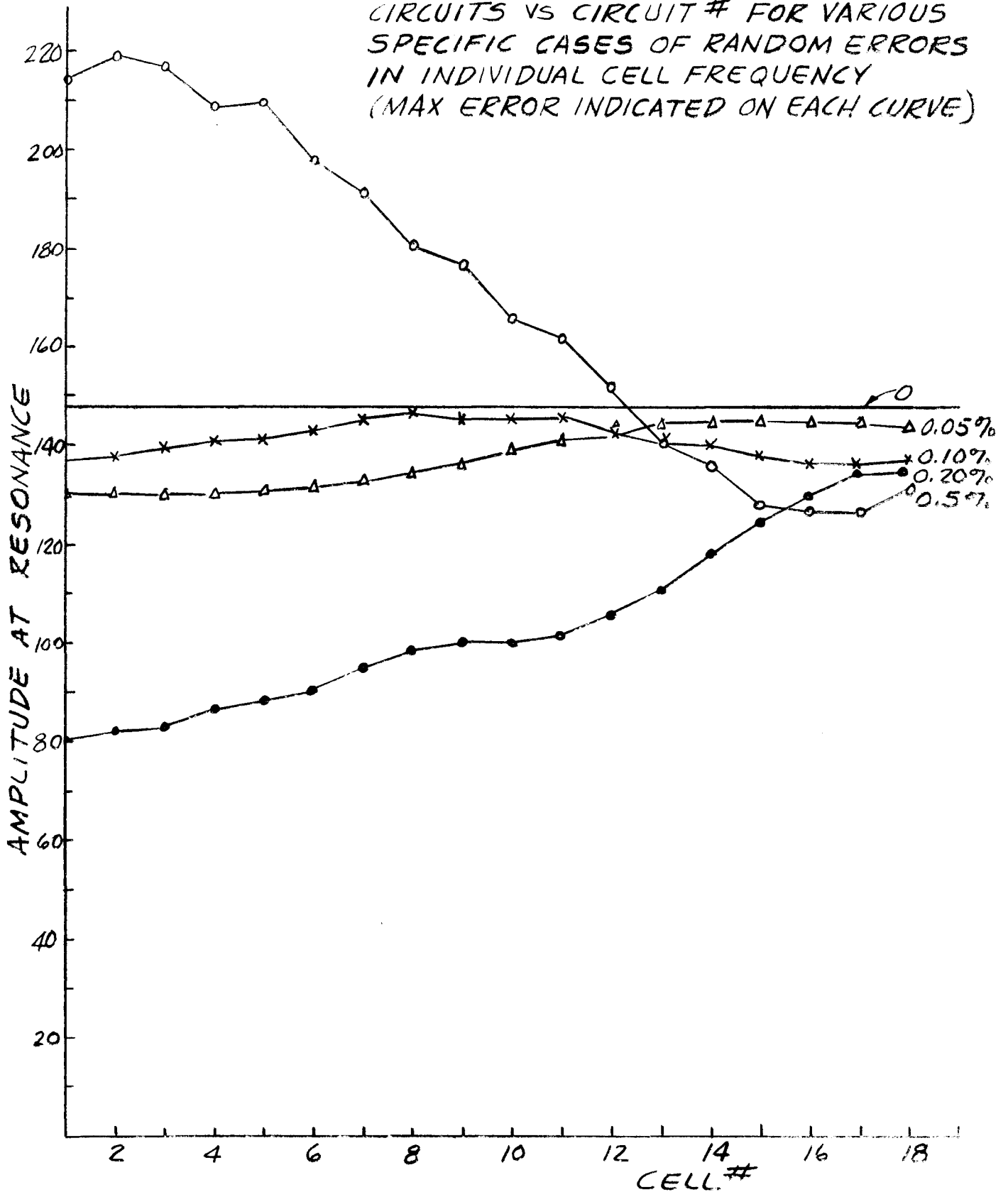


FIG. 6-AMPLITUDE OF CURRENT IN CIRCUITS VS CIRCUIT # FOR VARIOUS SPECIFIC CASES OF RANDOM ERRORS IN INDIVIDUAL CELL FREQUENCY (MAX ERROR INDICATED ON EACH CURVE)





how it shall be done, because of the interrelation. A Fourier amplitude of the current in each cell could be taken, for example. This might not be feasible, but it is possible in principle.

The effect of changing the coupling constant,  $k$ , on the error in circuit amplitude is shown in Fig. 7 (on the same circuit presented in Fig. 5 -- randomly distributed values of  $\Delta\omega \approx 10^{-3}$  to  $10^{-4} \omega_0$ ). You can see that it varies roughly as  $k^{-1}$ . Thus the stronger the coupling, the less the derivation from flatness, for a given error. In order to flatten such a system, one wants as broad a bandwidth as possible.

Figure 8 shows the maximum amplitude excursion as a function of the total number of cells in the system, for three different magnitudes of error, for a bandwidth of 25%. As the number of cells increases, so does the maximum amplitude error. Errors of  $10^{-6}$  to  $10^{-5}$  are pretty hard to achieve, but if the errors are an order of magnitude higher, the maximum error in current amplitude is of the order of 4% for 36 cells. This gives an idea of how good the tuning must be to get flatness.

(The discussion of this paper is included with that following the next paper.)

#### Reference

- (1) L. W. Alvarez, et al. Rev. Sci. Instr. 26, 111 (1955).

