TRANSVERSE BEAM BLOW-UP IN STANDING WAVE LINACS

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I. Introduction

The phenomenon of beam blow-up in traveling wave electron linear accelerators[1] has caused concern as one has attempted to increase the beam current and pulse length. Among the explanations of the observed phenomenon is one presented by Wilson[2] in analogy with the theory of backward wave oscillators for accelerating-type fields. This explanation suggests that means exist to reduce the serious consequences of the blow-up by modifying the synchronism of the beam and backward wave deflecting mode. The effect discussed by Wilson takes place for bunched or unbunched beams.

A similar concern has been expressed by Leiss and Schrack[3] who point out that a bunched beam may have a harmonic which resonates with that component of the deflecting mode traveling with the beam velocity. For conventional iris-loaded guides, the deflecting mode frequency band is approximately 50% above that for the accelerating mode. If the beam bunching is, for example, at the 4th subharmonic of the accelerating mode (as it is at present for the AGS improvement program), then the 6th harmonic of the beam will be approximately resonant with the deflecting mode. This phenomenon clearly depends on the details of the mode spectrum for different values of β and can probably be influenced by transverse focusing and by perturbation of the deflecting modes where necessary.

The present work is an attempt to formulate the two corresponding effects for the case of a standing wave accelerator (proton linac in the case of the AGS improvement program). We will not treat here the additional serious effect of beam loading on the accelerating mode. (In this case the steady state effect can presumably be compensated for by adjusting the power source to supply additional power at high beam currents; the transient effect will require beam injection while the fields are rising in an appropriate way.)

II. Cavity Fields

We shall consider an iris-loaded guide of length $L = NL_o$, where L_o is the cell length, as shown in Fig. 1. The deflections will be assumed to be confined to the x - z plane, where z is taken as the longitudinal direction. In this plane the j^{th} normalized standing wave

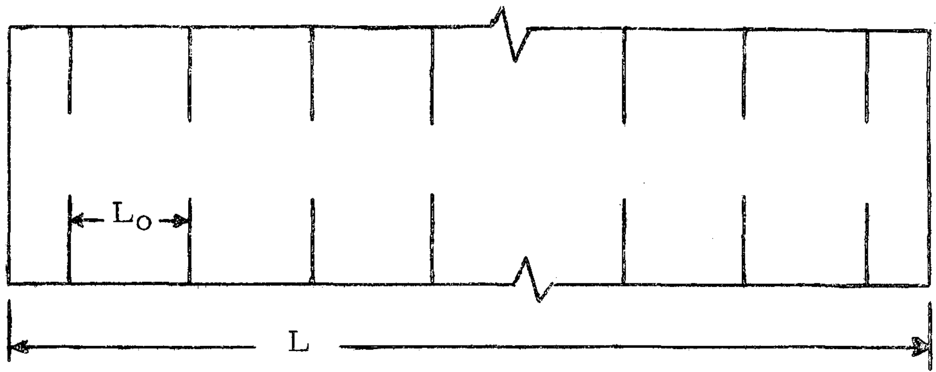

Fig. 1

deflecting mode can be written, because of Floquet's theorem, as

$$E_x^j\,(x, o, z) = \sum_{n=o}^{N-1} P_{jn}\,(x) \sin k_{jn} z\,,$$

$$H_y^j\,(x, o, z) = \sum Q_{jn}\,(x) \cos k_{jn} z\,, \qquad (2.1)$$

$$E_z^j\,(x, o, z) = \sum R_{jn}\,(x) \cos k_{jn} z\,,$$

where

$$k_{jn} = k_j + 2\,n\,\pi / L_o\,, \qquad k_j = j\,\pi / L\,. \qquad (2.2)$$

If we define the space harmonic of concern as that for $n = 0$, the important components of the fields in the vicinity of the axis may be written as

$$E_x^j \simeq P_j \sin k_j z$$
$$H_y^j \simeq Q_j \cos k_j z \qquad (2.3)$$
$$E_z^j \simeq x\,R_j \cos k_j z\,.$$

The fields are normalized such that

$$\nabla \times \vec{E}^j = (\omega_j/c)\,\vec{H}^j\,, \qquad \nabla \times \vec{H}^j = (\omega_j/c)\,\vec{E}^j\,,$$

$$\int dv\,\vec{E}^i \cdot \vec{E}^j = \int dv\,\vec{H}^i \cdot \vec{H}^j = \delta_{ij}\,, \tag{2.4}$$

and the actual fields are written in terms of the normalized fields as

$$\vec{E} = \sum_j C_j\,(t)\,E^j\,(\vec{x})\,, \qquad \vec{H} = \sum_j D_j\,(t)\,H^j\,(\vec{x})\,, \tag{2.5}$$

where C_j and D_j are related in the absence of beam by

$$D_j = (\epsilon c/\omega_j)\,\dot{C}_j\,, \qquad C_j = -(\mu c/\omega_j)\,\dot{D}_j\,. \tag{2.6}$$

Relations can be obtained between P_j, Q_j, R_j via Maxwell's equations. One must, however, take into account the component H_z in obtaining these relations, since the modes are not purely TE or TM, but are hybrid modes.[4] Let us write:

$$Q_j = F_j R_j\,, \qquad P_j = K_j R_j\,. \tag{2.7}$$

In the presence of a current pulse of the form

$$\vec{J}\,(\vec{x}, t) = \int_{-\infty}^{\infty} \vec{j}\,(\vec{x}, \omega)\,e^{i\omega t}\,d\omega \tag{2.8}$$

it can be shown that $C_j\,(t)$ satisfies a differential equation whose solution can be written in the form

$$C_j\,(t) = \frac{-i}{\epsilon}\int d\omega\,\frac{\omega\,e^{i\omega t}}{\omega^2 - \omega_j^2}\int \vec{j}\,(\vec{x}, \omega)\cdot E^j\,(\vec{x})\,dv\,. \tag{2.9}$$

The poles are moved into the upper half ω-plane because of losses. Assuming the losses to be small, one can write for late times (after the beam pulse has passed through the cavity)

$$C_j\,(t) = -(\pi/\epsilon)\left[e^{i\omega_j t}\int \vec{j}\,(\vec{x}, \omega_j)\cdot \vec{E}^j\,dv + c.c.\right]$$

$$\equiv -(\epsilon c)^{-1}\,D_j\cos(\omega_j t + \phi_j)$$

$$D_j\,(t) = -\pi c\left[i\,e^{i\omega_j t}\int \vec{j}\,(\vec{x}, \omega_j)\cdot \vec{E}^j\,dv + c.c.\right] \tag{2.10}$$

$$\equiv D_j\sin(\omega_j t + \phi_j)\,.$$

In the presence of losses these fields will decay at the rate

$$e^{-\frac{\omega_j}{2 Q_j} t} \quad . \tag{2.11}$$

The transverse angular impulse given to a particle entering the cavity at $t = t_m$ in the presence of deflecting fields can be calculated in the approximation that the trajectory is along the longitudinal axis. One then has

$$\Delta \theta_x = (e/pv) \int_0^L dz \; \left[E_x - \mu \, v \, H_y\right]_{t = t_m + z/v}$$

$$= (e/\epsilon pc) \sum_j \omega_j^{-1} \int_0^L dz \; R_j \cos k_j z \; D_j \sin(\omega_j z/v + \psi_j) \, , \tag{2.12}$$

where

$$\psi_j = \omega_j t_m + \phi_j \, . \tag{2.13}$$

Since one is interested only in those modes for which k_j and ω_j/v are almost equal (phase velocity of deflecting mode approximately the same as beam velocity), one may write

$$\Delta \theta_x \cong (eL/\epsilon pc) \sum_j (\omega_j \alpha_j)^{-1} D_j R_j \sin(\alpha_j/2) \sin(\psi_j - \alpha_j/2) \, , \tag{2.14}$$

where

$$\alpha_j \equiv (k_j - \omega_j/v) L \tag{2.15}$$

is the slip of beam relative to the wave in its transit through the cavity. If one adds angular impulses due to all effects, the beam will be lost when

$$\Lambda_x \Delta \theta_x \sim 2 \pi a \, , \tag{2.16}$$

where Λ_x is the wavelength of the transverse oscillation and a is the bore radius.

The remaining quantity needed for our analysis is $\vec{j}(\omega)$. Assuming a single bunch of spatial dependence f (z) (symmetric about z = 0 for convenience) whose center follows the trajectory $z = vt$, $x = x(t)$, one has

$$\vec{J}(\vec{x}, t) = I_o \Delta t\, \delta\left[x - x(t)\right] \delta(y) f\left[z - v(t - t_m)\right] \left[\vec{i}\, \dot{x}(t) + \vec{k}\, v\right]$$

where $\Delta t = 2\pi/\omega_o$ is the separation of beam bunches and I_o is the average current. Writing

$$f(u) = \frac{1}{2\pi v}\int d\omega\; e^{-i\frac{\omega}{v}u}\, g(\omega), \quad g(\omega) = \int_{-\infty}^{\infty} du\; e^{i\frac{\omega}{v}u}\, f(u) \qquad (2.17)$$

one finds, for narrow bunches,

$$\vec{j}(\vec{x}, \omega) = \frac{I_o}{\omega_o}\, g(\omega)\, \delta\left[x - x(t_m + z/v)\right] \delta(y)\left[\vec{i}\, \dot{x}(t_m + z/v)/v + \vec{k}\right] e^{-i\omega t_m - i\omega z/v} \qquad (2.18)$$

The normalization is such that for a delta function bunch $g(\omega) = 1$; that is, $g(\omega)$ is the relative harmonic content of the beam pulse. The relevant integral in (2.10) is therefore

$$\int \vec{j}(\vec{x}, \omega_j) \cdot \vec{E}^j\, dv =$$

$$(I_o/\omega_o v)\, g(\omega_j)\, e^{-i\omega_j z/v}\left[\dot{x}(t_m + z/v)\, E_x^j(x(t_m + z/v), 0, z) + \right.$$

$$\left. + v\, E_z^j(x(t_m + z/v), 0, z)\right] \qquad (2.19)$$

In the limit of small transverse displacements, one then finds from (2.3), (2.7):

$$\int \vec{j}(\vec{x}, \omega_j) \cdot \vec{E}^j\, dv =$$

$$(I_o/\omega_o)\, g(\omega_j)\, e^{-i\omega_j t_m}\, R_j \int_o^L dz\; e^{-i\omega_j z/v}\left[K_j(\dot{x}/v)\sin k_j z + \right.$$

$$\left. + x\cos k_j z\right] \simeq (I_o/2\omega_o)\, g(\omega_j)\, e^{-i\omega_j t_m}\, R_j \int_o^L dz\; e^{i\alpha_j z/L}$$

$$\left[x(z) - K_j\, i x'(z)\right] \quad . \qquad (2.20)$$

III. Nonresonant Beam Blow-up

If one considers that the beam has transverse displacement and velocity unaffected by the cavity fields in calculating the contribution to (2.20), then the contribution of successive bunches will depend on their relative phase, and build-up will be serious only if there is a resonance between a beam pulse harmonic and a defelcting mode. This phenomenon is treated in Section IV. If no resonance exists, one must consider the effect of the field itself on the trajectory in order to get a significant contribution to (2.20). This is the effect considered by Wilson, and the one we shall treat in Section III for a standing wave linac.

In order to evaluate (2.20), we must find the trajectory of that particle entering the cavity which is oscillating with fields

$$\begin{aligned} C_j(t) &= -(\epsilon c)^{-1} D_j \cos(\omega_j t + \phi_j) \\ D_j(t) &= D_j \sin(\omega_j t + \phi_j), \end{aligned} \qquad (3.1)$$

at the time t_m. According to (2.12) one has

$$\Delta\theta_x = x'(z) = (e/pv)\int_0^z dz\left[E_x - \mu v H_y\right]_{t = t_m + z/v} . \qquad (3.2)$$

Keeping only those wave components traveling at approximately the same velocity as the particle, one finds

$$x'(z) \simeq -\sum_k (e/\epsilon p c \omega_k)\left\{\left[D_k(t_m + z/v) E_x^k\right]_0^z - \int_0^z dz\, D_k(t_m + z/v)(\partial E_z^k/\partial x)\right\}$$

$$\simeq (e/2\epsilon pc)\sum_k (D_k R_k/\omega_k)(L/\alpha_k - K_k)\left[\cos(\alpha_k z/L - \psi_k) - \cos\psi_k\right], \qquad (3.3)$$

$$x(z) = (e/2\epsilon pc)\sum_k (D_k R_k/\omega_k)(L/\alpha_k - K_k)(L/\alpha_k)\left[\sin(\alpha_k z/L - \psi_k) + \sin\psi_k - (\alpha_k z/L)\cos\psi_k\right], \qquad (3.4)$$

where

$$\psi_k = \omega_k t_m + \phi_k . \qquad (3.5)$$

The nonresonant character of the values of t_m implies that only the $e^{i\psi_k}$ component of (3.3) and (3.4) is needed with the factor $e^{-i\omega_j t_m}$ in (2.20). In this approximation one has

$$\int \vec{j} \cdot \vec{E}^j \, dv = (e I_o / 8 \omega_o \epsilon \, pc) \, g(\omega_j) \sum_k e^{i(\omega_k - \omega_j) t_m + i \phi_k} (D_k R_j R_k / \omega_k)(L/\alpha_k - K_k)$$

$$\times \int_o^L dz \, e^{i\alpha_j z/L} \left[(L/\alpha_k)(i \, e^{-i\alpha_k z/L} - i - \alpha_k z/L) - K_j \, i \, (e^{-i\alpha_k z/L} - 1) \right]$$

$$\equiv \sum_k M_{jk} \, e^{i m_{jk}} \, e^{i(\omega_k - \omega_j) t_m} \, D_k \, e^{i \phi_k} , \qquad (3.6)$$

where M_{jk} and m_{jk} are the amplitude and phase of all the factors not appearing explicitly in the last form of (3.6). From (2.10), the pulse entering at t_m contributes an increment to the field already present, which is given, including the decay of the field already present, by

$$\Delta D_j(t) \cong 2\pi c \sum_k D_k M_{jk} \sin \left[\omega_j t + (\omega_k - \omega_j) t_m + \phi_k + m_{jk} \right]$$

$$- (\omega_j \Delta t / 2 Q_j) D_j \sin (\omega_j t + \phi_j) . \qquad (3.7)$$

This is our result. It is, of course, dependent on the mode spectrum (ω_j vs j), the timing of the pulses* (t_m), the slip (α_j vs j), and all the other nonexponential factors appearing in M_{jk}. Particularly in cases of low group velocity (ω_k close to ω_j), one should proceed directly from (3.7) on a numerical basis.

We shall try to reduce (3.7) analytically by making further assumptions. In particular we shall assume a steady state solution for which the contributions for $k \neq j$ in (3.7) average to zero. In this case

$$\Delta D_j(t) = 2\pi c \, M_{jj} \, D_j \sin (\omega_j t + \phi_j + m_{jj}) - (\pi \omega_j / \omega_o Q_j) D_j \sin (\omega_j t + \phi) , \qquad (3.8)$$

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*We have already assumed that the pulses are nonresonant and have ignored the term in $(\omega_k + \omega_j) t_m$. If they are resonant, the present considerations are modified by a factor of order 2.

which is equivalent to the relations

$$\frac{\Delta D_j}{D_j} = 2\pi c M_{jj} \cos m_{jj} - \frac{\pi \omega_j}{\omega_o Q_j} \qquad (3.9)$$

and

$$\Delta \phi_j = -2\pi c M_{jj} \sin m_{jj} . \qquad (3.10)$$

From (3.6), one finds

$$M_{jj} \cos m_{jj} = (e I_o / 8 \omega_o \epsilon pc) g(\omega_j)(R_j^2/\omega_j)(L/\alpha - K) L$$

$$\times \mathrm{Re} \int_0^1 dx \left[i (L/\alpha - K)(1 - e^{i\alpha x}) - \alpha x e^{i\alpha x} L/\alpha \right]$$

$$= (e I_o / 4 \omega_o \epsilon pc) g(\omega_j)(R_j^2/\omega_j)(L/\alpha - K) L/\alpha \left[L/\alpha (1 - \cos\alpha - \tfrac{\alpha}{2} \sin\alpha) - \right.$$

$$\left. - K(1 - \cos\alpha)/2 \right] . \qquad (3.11)$$

In the approximation of small coupling holes it can be shown that

$$\frac{L}{\alpha} \gg K . \qquad (3.12)$$

Equation (3.11) then becomes

$$M_{jj} \cos m_{jj} = (e I_o / 2\pi^3 \omega_o \epsilon pc) g(\omega_j)(R_j^2/\omega_j) L^3 g_2(\alpha_j) \qquad (3.13)$$

where

$$g_2(\alpha) = \frac{1 - \cos\alpha - (\alpha/2)\sin\alpha}{2(\alpha/\pi)^3} \qquad (3.14)$$

is the same function as that defined by Wilson, and has a maximum value of 1.04 when $\alpha = 2.65$.

The "starting current" for the beam blow-up is therefore given by the vanishing of (3.9), that is for

$$e I_o = \frac{\pi^3 \epsilon p \omega_j^2}{g_2 g L^3 R_j^2 Q_j} . \qquad (3.15)$$

The quantity $Q_j R_j^2$ is related to the ratio of the square of the electric field gradient along the axis to the power loss and therefore has the general form of a shunt resistance per unit length for the deflecting mode. Specifically, if one defines r_ℓ as

$$r_\ell = \frac{\left[(c/\omega_j)\int_o^L dz\ (\partial E_z^j/\partial x)\cos k_j z\right]^2}{L \times \text{Power Loss}} , \tag{3.16}$$

and notes that Q_j may be written as

$$Q_j = \frac{(\omega_j \epsilon/2)\int |\vec{E}^j|^2 dv}{\text{Power Loss}} = (\omega_j \epsilon/2\ \text{Power Loss}) \tag{3.17}$$

one has

$$\frac{r_\ell}{Q_j} = L R_j^2 (c/\omega_j)^3 Z_o/2 , \tag{3.18}$$

where

$$Z_o = \sqrt{\mu/\epsilon} = 377 \text{ ohms} . \tag{3.19}$$

This leads finally to

$$e I_o = \frac{\pi^3 Mc^2 (c/\omega_j)\beta\gamma}{2 g_2 g r_\ell L^2} . \tag{3.20}$$

As an illustration of the order of magnitude of (3.20), we shall apply it to the first iris cavity in the proposed new AGS injector where we use the following parameters:

$$\begin{aligned} Mc^2 &= 940 \text{ Mev} \\ 2\pi c/\omega_j &= 0.25 \text{ m} \\ \beta &= 0.6 \\ \gamma &= 1.25 \\ g_2 \sim g &\sim 1 \\ L &\sim 3 \text{ m} \\ r_\ell &\sim 20 \text{ megohms/m} \end{aligned} \tag{3.21}$$

leading to

$$I_o \simeq 2 \text{ amp.} \tag{3.22}$$

Several comments should be made at this time:

1) The value of r_ℓ has been chosen typical of accelerating modes. It will probably be considerably smaller since the square of a transit time factor should be included to correspond to the particular space harmonic used. This will lead to an even higher "starting current."*

2) Resonance between the deflecting mode and the beam bunch frequency will lead to a reduction of order of a factor 2 in (3.20) and (3.22).

3) If the current is higher than the limit in (3.20), the build-up time can be estimated from (3.9) with an assumption for the order of magnitude of the "noise" in the cavity. The result will be of the form

$$T_B = \frac{\text{Const.}}{I_o - I_{o,\,\text{starting}}} . \tag{3.23}$$

4) Although the present calculation is for a standing wave linac, the form of Wilson's result for a traveling wave linac can be obtained by going to the limit of large L in (3.6). The sum over k becomes an integral over k and a sum over beam pulses (m) leads to

$$\sum_m e^{i(\omega_k - \omega_j) m \Delta t} = 2\pi\, \delta\left[(\omega_k - \omega_j)\ \Delta t\right]$$

$$\cong \left(\frac{d\pi}{dk}\right)^{-1} \frac{\omega_o L}{\pi}\ \delta(k - j) = \frac{\omega_o L}{\pi v_g}\ \delta(k - j) . \tag{3.24}$$

The factor M_{jj} in (3.6) must then be multiplied by the factor $(\omega_o L/\pi v_g)$ and the self-consistent condition $\Delta D/D = 1$ leads to (3.20) multiplied by the factor $v_g Q_j/\omega_j L$. This factor, which is the ratio of the decay time to the filling time, is not surprising when one goes from a standing to a traveling wave linac. The net result for the starting current is an expression of the form

$$e\, I_o^{T.W.} \sim \frac{Mc^2 \beta\gamma\, (v_g/c)\, (c/\omega)^2}{g_2\, g\, L^3} \left(\frac{Q}{r_\ell}\right) , \tag{3.25}$$

which is identical in its dependence on the parameters with Wilson's result.

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*An estimate of r_ℓ has been made for a model of independent cells. This leads to $r_\ell \sim 1$ megohm/m and gives a starting current of 40 amp. See Appendix.

5) The contributions from $k \neq j$ in (3.7) may not be negligible, as indicated by the traveling wave result. However, the factor M_{jk} has a structure which confines contributions to the vicinity $\alpha_j \sim \alpha_k \sim \pi$. An estimate of the effect of the term

$$(\omega_k - \omega_j)\, t_m \qquad (3.26)$$

for an adjacent mode can be made by taking t_m to be the build-up time, which is assumed to be of the same order as the decay time, i.e.,

$$t_m \sim Q/\omega_j \,. \qquad (3.27)$$

For this choice

$$(\omega_k - \omega_j)\, t_m \sim v_g\, Q_j/\omega_j\, L \,. \qquad (3.28)$$

The contributions will therefore be confined to the mode $k = j$ or will extend to the neighboring modes according to whether the parameter in (3.28) is greater than or less than 1. If it were not for the factor M_{jk}, one would then multiply (3.20) by the factor $v_g\, Q/\omega_j\, L$ as we did in (3.25) for the traveling wave case.

6) Proper numerical investigation of this phenomenon should be performed starting with (3.7), with an appropriate deflecting mode spectrum.

7) Motion in the y direction has been neglected. If this is taken into account, the cavity oscillations may be induced with rotating polarization. This can, of course, be prevented by destroying the azimuthal symmetry.

IV. Resonant Beam Blow-up

The other possible serious effect previously mentioned occurs if one of the harmonics of the beam frequency resonates with one of the deflecting modes. In this case the transverse motion of the bunches induces cavity fields which may then build up from successive pulses. Since the build-up leads to deflections, this phenomenon involves not only the transverse focusing system, but the entire transverse history of the beam in preceding cavities.

According to (2.20), the m^{th} beam bunch, traversing a cavity with average transverse displacement and angle given by x_m and x'_m, gives a contribution to the current of

$$\int \vec{j}\,(\vec{x}, \omega_j) \cdot \vec{E}^j \, dv =$$

$$= (I_o\, g/2\,\omega_o)\, R_j\, (L/\alpha)\, e^{-i\omega_j t_m}\, (x_m - K_j\, ix'_m)(e^{i\alpha} - 1)\,. \quad (4.1)$$

The contribution to the field is therefore

$$\Delta D_j\,(t) = (\pi c\, I_o\, g\, L\, R_j\, g_3/\omega_o)\, \mathrm{Re}\left\{ e^{i\omega_j(t - t_m) + i\alpha/2} (x_m - K_j\, ix'_m) \right\}\,. \quad (4.2)$$

The factor $g_3 = \frac{\sin(\alpha/2)}{\alpha/2}$ indicates that the effect is appreciable only for those deflecting modes with phase velocities close to the beam velocity. The factor $e^{-i\omega_j t_m}$ indicates that the contributions of successive bunches will be out of phase unless

$$\omega_j\, \Delta t/2\pi = \omega_j/\omega_o \simeq s\,, \quad (4.3)$$

that is, unless the ratio of deflecting mode frequency to beam frequency is close to an integer.

The angular deflection due to the presence of deflecting modes is given by (2.14). If one also includes the effects of the focusing forces, the coupled set of equations (2.14) and (4.2) allow one to follow the progress of the effect from cavity to cavity and from pulse to pulse numerically. [The effect treated in Section III may even be included by adding (3.8) to (4.2).] It is clear that in this case the effect depends even more sensitively on the deflecting mode spectrum.

An order of magnitude estimate of the effect can be made in the steady state condition (constant values of x_m, x'_m from pulse to pulse) by including the field decay due to losses. The field after the entrance of the n^{th} bunch is then

$$D_j\,(t) = (\pi c\, I_o\, g\, L\, R_j\, g_3/\omega_o)\, \mathrm{Re}\left\{ e^{i\omega_j(t - t_n) + i\alpha/2} (x - K_j\, ix') \sum_{m=o}^{\infty} e^{2\pi m\left[i(\omega_j/\omega_o - s) - (\omega_j/2\omega_o Q_j\right]} \right\}$$

$$\simeq (c\, I_o\, g\, L\, R_j\, g_3/2\,\omega_o)\, \mathrm{Re}\left\{ \frac{e^{i\omega_j(t - t_n) + i\alpha/2} (x - K_j\, ix')}{(\omega_j/2\,\omega_o Q_j) - i\,(\omega_j/\omega_o - s)} \right\}. \quad (4.4)$$

Since Q_j will normally be much larger than $\left| \omega_j/\omega_o - s \right|$, (4.4) can be reduced to

$$D_j\,(t) \simeq (\pi c\, I_o\, g\, L\, R_j\, g_3) \left[\frac{K_j\, x' \cos \chi_j - x \sin \chi_j}{\omega_j - s\,\omega_o} \right], \qquad (4.5)$$

where

$$\chi_j = \omega_j\,(t - t_n) + \alpha/2\;. \qquad (4.6)$$

The deflection of the n^{th} bunch in this field is given by (2.14). The $\cos \chi_j$ term in (4.5) corresponds to $\psi_j = \alpha_j/2 + \pi/2$ in (2.13), while the $\sin \chi_j$ term corresponds to $\psi_j = \alpha_j/2$. One therefore has, using (3.18),

$$\Delta x' \simeq x' \sum_j (e\,\pi\, I_o\, g\, g_3^2\, L/p)(r_\ell/Q_j)/(\omega_j - s\,\omega_o)\, K_j\, (\omega_j/c)^2\;. (4.7)$$

The increase in transverse amplitude due to this angular deflection is

$$\Delta A_x \simeq \sum_j \frac{\Lambda_x\, e\, I_o\, g\, g_3^2\, L\, \omega_j\, r_\ell\, K_j\, (\omega_j/c)}{2\, Mc^2 \beta\, \gamma\, Q_j\, (\omega_j - s\,\omega_o)}\;. \qquad (4.8)$$

For the independent cell model used before, and the AGS improvement program parameters, including I_o = 100 mA, one finds (see Appendix)

$$\Delta A_x \sim \frac{3 \times 10^{-7}}{(1 - \frac{s\,\omega_o}{\omega_j})} \text{ meters}\;. \qquad (4.9)$$

Thus, for a resonance accurate to 10^{-3}, the increase in transverse amplitude from this cavity is ~ 0.03 cm. Since it is unlikely that this resonance will persist to an accuracy of 10^{-3} for several cavities, the effect appears not to be serious. If it should turn out that the resonance is more serious in one of the cavities, a perturbation of the deflecting modes of order 10^{-3} can undoubtedly be readily provided for.

In summary, therefore, it appears that the transverse motion of the beam can lead to a build-up of the deflecting mode if there is a resonance between the beam and deflecting mode frequencies. A rough estimate of the effect indicates that it is small, but numerical estimates using (3.8), (4.2) and (2.14) are desirable. For these and other reasons it is recommended that both the accelerating and deflecting mode spectrum be measured carefully at several values of β.

We have of course neglected the variation of cell geometry within a cavity. This should reduce the effects discussed in this paper even further.

V. Summary

We have calculated the build-up of deflecting modes due to two separate causes. The first (Section III) is the nonresonant effect that comes from the deflection of the beam by the transverse fields already present. These deflections induce further transverse fields which have a component in phase with the original field. The appropriate equation governing the behavior of a bunched beam due to this effect is (3.7), with M_{jk} and m_{jk} defined in (3.6). For an approximate treatment of the steady state behavior, (3.20) is relevant. A further approximate calculation of r_ℓ is contained in the Appendix, and the corresponding current limit is given in (A-12).

The second effect treated (Section IV) is the resonant build-up of the deflecting modes due to the transverse oscillations of the beam. The average displacement and angle of a beam bunch with respect to the longitudinal cavity axis induces a transverse field. Successive bunches can build up this field if there is a resonance between the transverse mode and one of the harmonics of the beam. The appropriate equations governing the build-up of these coupled "displacement-field" oscillations are (4.2) and (2.14). For an approximate treatment of the steady state behavior, (4.5), (4.7) and (4.8) are relevant. A further approximate calculation of r_ℓ / Q_j is contained in the Appendix, and the corresponding limit is given in (A-10).

Numerical computation of the combination of the above two effects can also be performed. For this purpose the field increment per pulse is the sum of (3.8) and (4.2).

It is clear that both effects limit the contributions to those modes which slip no more than 180° behind the beam. In addition, the second effect is significant only if the resonance exists. For these reasons it is important to measure the mode spectrum for the cavities accurately for both the accelerating and deflecting bands at several values of β.

Estimates of the magnitudes of the two effects discussed have been made for the AGS improvement program parameters. Although these are extremely crude, the values obtained are not at all serious and should not prove troublesome.

We have not treated the case of two transverse dimensions, nor have we considered the effects of the induced fields on the bunching of the beam pulses. Moreover, we have treated each cavity as having a constant value of v and a uniform cell geometry. These effects will hopefully not be serious. Besides, one always has the possibility of perturbing the transverse modes to modify transverse effects. We have also not considered the possibility of longitudinal beam blow-up.

VI. Acknowledgements

It is a pleasure to acknowledge several helpful conversations with Dr. T. Nishikawa of the University of Tokyo, and with Dr. W. Walkinshaw of the Rutherford Laboratory.

APPENDIX

We shall estimate r_ℓ and Q for the deflecting modes under the assumption that the cells are approximately uncoupled. In this case the fields in the m^{th} cell are TM_{110} of the form

$$E_z^j = A_m^j \, J_1 \, (p_{11} \, r/b) \cos \theta$$

$$H_\varphi^j = A_m^j \, J_1' \, (p_{11} \, r/b) \cos \theta \qquad \text{(A-1)}$$

$$H_r^j = A_m \, \frac{J_1 \, (p_{11} \, r/b)}{p_{11} \, r/b} \sin \theta \,,$$

where

$$A_m^j = A_o^j \cos (m \, k_j \, L_o) \,, \qquad b = p_{11} \, c/\omega_j \,, \qquad \text{(A-2)}$$

and p_{11} is the first zero of J_1. The Fourier decomposition of E_z^j can be shown to be

$$E_z^j = J_1 \, (p_{11} \, r/b) \cos \theta \, A_o^j \sum_{n=o}^{N-1} \frac{\sin k_{jn} \, L_o}{k_{jn} \, L_o} \cos k_{jn} \, z, \qquad \text{(A-3)}$$

where k_{jn} is given by (2.2). The relevant deflecting mode (n = 0) in the vicinity of the axis is therefore given by

$$E_z^j = x \, (p_{11} \, A_o^j \, g_1 / 2 \, b) \cos k_j \, z \,. \qquad \text{(A-4)}$$

Comparison with (2.3) indicates that

$$R_j = \omega_j \, A_o^j \, g_1 / 2 \, c \,, \qquad \text{(A-5)}$$

where

$$g_1 = \frac{\sin k_j \, L_o}{k_j \, L_o} \,. \qquad \text{(A-6)}$$

The quantity A_o^j is obtained from the normalization condition (2.4). It can be shown to be

$$A_o^{j2} = \left[4 \, \omega_j^2 / \pi \, L \, p_{11}^2 \, c^2 \, J_o^2 \, (p_{11}) \right] \simeq \left[2 \, \omega_j^2 / L \, p_{11} \, c^2 \right] \,. \qquad \text{(A-7)}$$

From (3.18) one has

$$\frac{r_\ell}{Q_j} = L R_j^2 (c/\omega_j)^3 Z_o/2 \simeq \frac{Z_o g_1^2 (\omega_j/c)}{4 p_{11}} . \qquad \text{(A-8)}$$

For small holes one can show that

$$K \omega/c \sim (p_{11}^2/4\beta)(a/b)^2 \qquad \text{(A-9)}$$

where a is the hole radius. Equation (4.8) can then be written as

$$\Delta A_x \simeq \sum_j \frac{A_x e I_o Z_o L g g_1^2 g_3^2 (\omega_j/c) p_{11} a^2}{32 Mc^2 (1 - s \omega_o/\omega_j) \beta^2 \gamma b^2} . \qquad \text{(A-10)}$$

In order to obtain r_ℓ or Q_j independently, one must calculate the power loss. This is obtained from the square of the tangential magnetic field on both the guide and cell walls, and turns out to be

$$r_\ell \simeq \left(\frac{Z_o}{\delta}\right)\left(\frac{g_1^2}{8}\right)\left(\frac{L_o}{L_o + b}\right) , \quad Q_j^{-1} = \frac{2 \delta(\omega_j/c)}{p_{11}} \left(\frac{L_o + b}{L_o}\right) \qquad \text{(A-11)}$$

where δ is the skin depth. Equation (3.20) then becomes

$$e I_o = \left(\frac{4 \pi^3}{g g_1^2 g_2}\right)\left[\frac{\delta \beta \gamma Mc^2 (c/\omega_j)}{Z_o L^2}\right]\left(\frac{L_o + b}{L_o}\right) . \qquad \text{(A-12)}$$

LEISS: Does not the resonant beam blow-up always contribute in a serious way?

GLUCKSTERN: It does in the sense that the nonaxial component of the beam current can be increased by the response of the cavity. However, this depends on a resonance between the transverse mode and a multiple of the beam frequency. In addition the presence of transverse forces may modify the build-up.

E. KNAPP: Isn't the value of shunt impedance used (20 Megohms/meter) high?

GLUCKSTERN: Yes. As a first conservative calculation I took a value comparable with the accelerating mode. However, for the transverse one should take the appropriate space harmonic with a phase velocity equal to that of the particle. In this case the "starting" current is increased by a factor of order 10.

WALKINSHAW: Shouldn't one include other field components for the transverse modes?

GLUCKSTERN: Yes. Although the only components which do not vanish in the x-z plane are the ones used, the relation between these components depend on the other components--in particular, on $\partial H_z / \partial y$. This has only been taken into account approximately in the present calculation.

LEISS: Isn't it true that both effects (resonant and nonresonant beam blow-up) are present together and are part of the same effect?

GLUCKSTERN: I do not believe the two effects are the same, but both effects should be taken together in any proper calculation.

LEISS: Aren't there important resonant effects in your calculation of nonresonant beam blow-up?

GLUCKSTERN: I don't believe that these modify the results by more than a factor 2.

REFERENCES

1. M. C. Crowley-Milling et al., Nature 191, 483 (1961).

2. P. B. Wilson, HEPL Report No. 297, Stanford.

3. J. E. Leiss and R. A. Schrack, "Transient and Beam-Loading Phenomena in Linear Electron Accelerators", NBS Internal Report, October 30, 1962.

4. See, for example, H. Hahn, Rev. Sci. Instr. 34, 1094 (1963).