AN APPROACH TO THE STUDY OF BEAM LOADING FOR THE LINEAR ACCELERATOR

T. Nishikawa*
Brookhaven National Laboratory

1. Introduction

Most studies on the effects of beam loading in linear accelerators have been made using the power relations or the equivalent circuit analogies. On the other hand, the field equation for cavities and the equation of motion for the charged particles will give a more complete picture of this effect, especially when the accelerator is a standing wave type. This study was made for the iris-loaded structure being considered by the Brookhaven-Yale proton linac project.[1] The method is similar to the theories developed for microwave electronic devices, particularly traveling wave tubes, and the purpose is to give a brief physical picture but not exact numerical calculations.

2. Principle and Assumptions

According to Slater's well known theory for resonant cavities,[2] the solution of Maxwell's equations is expanded in terms of a summation over certain normal modes, which possess orthogonality properties. Any vector field can be broken into new fields, one of which is solenoidal and the other is irrotational and, so far as the wave propagation is considered, the space-charge effect can be separated for the first approximation. The effect of the beam current on the cavity field which is operated near the n^{th} mode is quite generally expressed by**

$$-\frac{1}{\epsilon\,\omega_{no}}\frac{\int \vec{J}\vec{F}_n^{\,*}dv}{\int \vec{E}\vec{F}_n^{\,*}dv} = j\left(\frac{\omega}{\omega_n} - \frac{\omega_n}{\omega}\right) + \frac{1}{Q_n} \quad . \tag{1}$$

$\vec{J}$ is the current density, $\vec{E}$ the field, and $\vec{F}_n$ is the solenoidal normal mode field. $\vec{F}_n$ is obtained by solving Maxwell's equations when there is no current in the cavity and the walls are either short-circuited or open-circuited. ω_{no} is the resonant angular frequency of the n^{th} mode and ω_n is that of the case when the effect of wall losses is taken into account. Rigorously $\omega_{no} \neq \omega_n$, but one can safely use ω_n instead of ω_{no} if the unloaded

---

* On leave from the Department of Physics, University of Tokyo.

** This equation is slightly modified from Slater's and immediately derived from the wave equation for the vector potential, and is quite general for the relativistic case. See Ref. 9 and its reference.

Q, Q_{on}, is sufficiently high. The total Q value of the cavity without beam, Q_n, is expressed by

$$\frac{1}{Q_n} = \frac{1}{Q_{on}} + \sum_i \frac{g_i}{Q_{ext,\ n,\ i}} , \tag{2}$$

where $Q_{ext,\ n,\ i}$ is the external Q to the i^{th} output and g_i is the real part of the reduced admittance of the i^{th} guide. If we study a standing wave type accelerator such as those operating in the π-mode, one section having N cells can be considered as a closed cavity of $Q_n = Q_{on}$. For simplicity throughout this paper a uniform cell structure in each cavity will be assumed, while an actual π-mode cavity would have nonuniform structure as discussed in the Appendix. The actual field, $\vec{E}$, is also assumed to vary as $e^{j\omega t}$ with the operating frequency ω, and the integration is performed over the cavity volume.

The effect of beam loading in a linac can be estimated from the left-hand side of (1); its imaginary part gives the shift of the resonant frequency or the detuning effect and the real part gives the further power loss or the beam Q-value.

3. Effect of Tightly Bunched Beam (Light-loading)

The assumption throughout this paper is that a tightly bunched beam interacts with the field propagating along the cavity axis. The beam is also assumed to be confined to a region having a small diameter about this axis (z axis).

In the first step, we neglect the field induced by the beam and the reaction of the field on the beam. This approximation will be valid when the loading is sufficiently small or in a transient stage when the induced field has not yet built up.

In terms of the field equation the stored energy for the n^{th} mode may be written as

$$W_n = \frac{1}{2} \epsilon \int \vec{E}\vec{F}_n^{\,*} dv \int \vec{E}^{*}\vec{F}_n dv . \tag{3}$$

This equation may also be written in terms of circuit parameters as

$$\omega_n W_n = Q_{on} R \quad L = \frac{Q_{on} E_{no}{}^2 L}{r_{sn}} , \tag{4}$$

where R is the wall loss per unit length, L the cavity length, E_{no} the axial-field amplitude (peak value of the field strength) of the n^{th} harmonic field, and $r_{sn} = E_{no}{}^2/R$ the shunt impedance for the n^{th} mode field. Substituting (3) and (4) into (1), we obtain

$$- \int \vec{J}\vec{F}_n{}^* dv \int \vec{E}^* \vec{F}_n dv = \frac{2Q_{on}}{r_{sn}} \; E_{no}{}^2 L \left[j \left(\frac{\omega}{\omega_n} - \frac{\omega_n}{\omega} \right) + \frac{1}{Q_n} \right] \tag{5}$$

Now the beam is assumed to be bunched in a wave having the propagation exponent $j(\omega' t - k' z)$ with a bunch width $\delta\phi$ at a center phase ϕ_o. The current density of such beam is analyzed in a Fourier series as

$$J = J_o + \mathrm{Re} \sum_{\nu=1}^{\infty} 4J_o \frac{\sin \frac{\nu \delta\phi}{2}}{\nu \; \delta\phi} \; e^{j\nu(\omega' t - k' z + \phi_o)} \tag{6}$$

where J_o is the mean density during one pulse of the linac operation. When $\nu \delta\phi << 1$, the ν component is

$$J_\nu \doteq 2J_o \; \mathrm{Re} \; e^{j\nu(\omega' t - k' z + \phi_o)} \tag{6'}$$

So far as a steady state solution is considered, only the term of $\omega_b = \nu\omega' \approx \omega_n$ * should be important in the estimation of integral $\int \vec{J}\vec{F}_n{}^* dv$. Taking $k_b = \nu k'$ and $\phi_b = \nu \phi_o$, it becomes

---

* ν is four for the proton linac planned by Brookhaven.

$$\int \vec{J}\vec{F}_n^{\,*} dv = -2jI_o F_{no} e^{j\phi_b} \left\{ \frac{1}{k_n + k_b} \left[1 - e^{-j(k_n + k_b)L} \right] \right. \tag{7}$$

$$\left. + \frac{1}{k_n - k_b} \left[1 - e^{-j(k_n - k_b)L} \right] \right\},$$

where I_o is the mean current, F_{no} is the axial-field amplitude of the n^{th} normal mode field which is given by $2F_{no} \cos k_n z$ in the standing wave case. For a π -mode, $k_n L = \pi N$ with the total number of cells, N, so that $k_n L >> 1$. In such a case, one can easily see that the right hand side of (7) is dominant for $|k_n - k_b| \approx 0$, or for the synchronized wave which hereafter is designated by n. It also should be pointed out that from here on only the synchronized primary mode will be considered, and the effects of all other modes have been neglected. Taking $|k_n - k_b| L << 1$, we get from (7),

$$\int \vec{J}\vec{F}_n^{\,*} dv \approx 2I_o F_{no} L e^{j\phi_b} . \tag{8}$$

Combining (5) and (8),

$$I_o e^{j\phi_{bn}} = - \frac{Q_{on}}{r_{sn}} E_{no} \left[j \left(\frac{\omega}{\omega_n} - \frac{\omega_n}{\omega} \right) + \frac{1}{Q_n} \right], \tag{9}$$

where we used a relation, $\vec{E} = \sum \vec{F}_n \int \vec{E}\vec{F}_n^{\,*} dv = \sum \vec{E}_n$ for the n^{th} harmonic field $\vec{E}_n$; and ϕ_{bn} is the relative phase of the beam bunch to the n^{th} harmonic field. Then we have the frequency shift, $\Delta\omega_n = \omega - \omega_n$, and the beam Q-value, Q_{bn}, as

$$\frac{2\,\Delta\omega_n}{\omega_n} = -\frac{r_{sn}}{Q_{on}}\,\frac{I_o}{E_{no}}\,\sin\phi_{bn} \tag{10a}$$

$$\frac{1}{Q_{bn}} = \frac{r_{sn}\,I_o}{Q_{on}\,E_{no}}\,\cos\,\phi_{bn}\ , \tag{10b}$$

where the total Q-value is expressed by $\frac{1}{Q_{tn}} = \frac{1}{Q_n} + \frac{1}{Q_{bn}}$.

4. Effect of Induced Field due to Beam Loading

If the beam loading increases, the field induced by the beam increases and becomes an appreciable part of the field, so that it should not be neglected. The actual field in a cavity can be divided into two parts; one is excited by the external source, $\vec{E}_e$, and the other is induced by the beam, $\vec{E}_b$. The total field is given by

$$\vec{E}_t = \vec{E}_e + \vec{E}_b \tag{11}$$

The induced field is also given by the use of (1), which is the so-called circuit equation in the present paper. The n^{th} harmonic field of E_b is

$$\vec{E}_{bn} = \vec{F}_n \int \vec{E}_b\,\vec{F}_n^{\,*} dv \tag{12}$$

and its axial component will be given by

$$\dot{E}_{bno} = F_{no} \int \vec{E}_b\,\vec{F}_n^{\,*} dv \tag{12'}$$

where $\dot{E}_{bno}$ is the complex amplitude including the phase. The axial-field amplitude of the n^{th} normal mode, F_{no}, which is, of course, a real number, is obtained from (3) and (4), where $\vec{E}$ is taken by $\vec{F}_n$. Since $W_n = \frac{\epsilon}{2}$ for the normal mode,

$$F_{no} = \sqrt{\frac{\epsilon}{2}\,\frac{r_{sn}\;\omega_n}{Q_{on}\;L}}\;. \tag{13}$$

The value of integral in (12′) is given as

$$\int \vec{E}_b\,\vec{F}_n^{\,*}dv = -\frac{\int \vec{J}\vec{F}_n^{\,*}dv}{\epsilon\,\omega_n\left[j\left(\frac{\omega_b}{\omega_n}-\frac{\omega_n}{\omega_b}\right)+\frac{1}{Q_n}\right]}$$

$$= -\sqrt{\frac{r_{sn}L}{2Q_{on}\epsilon\,\omega_n}}\;\frac{2I_o\,e^{j\phi_b}}{\left[j\left(\frac{\omega_n}{\omega_n}-\frac{\omega_n}{\omega_b}\right)+\frac{1}{Q_n}\right]} \tag{14}$$

from (1) and (8). Substituting (13) and (14) into (12′)

$$\dot{E}_{bno} = -\frac{r_{sn}}{Q_{on}}\;\frac{I_o\,e^{j\phi_b}}{\left[j\left(\frac{\omega_b}{\omega_n}-\frac{\omega_n}{\omega_b}\right)+\frac{1}{Q_n}\right]}\;, \tag{15}$$

just corresponding to (9).

In order to discuss the effect of induced field, we must consider the relation between ω_b and ω_n. The normal mode frequency, ω_n, is the proper frequency of the free oscillation of the cavity and is just the operating resonant frequency, ω_o, when beam is not present. If the beam comes in, the dynamic resonant frequency will move due to the reactive component of the load and differ from ω_n, the amount of this shift being given by (10a) in the first approximation. We can make a readjustment of the tuning of the cavity to resonate at the frequency ω_o, in order to keep the synchronous condition for the beam and the wave velocities. In such a case, the normal mode frequency is moved in turn by the tuning process and is no longer equal to ω_o. Because of the phase oscillation in the preceding part of the accelerator, the beam will be bunched to the wave of ω_o rather than that of ω_n, or $\omega_b \approx \omega_o \neq \omega_n$. Howevever, as we can see from (10a) and (10b), if $|\phi_b| << 1$ or $Q_b >> Q_{on}$, we may put $|\omega_b - \omega_n| << \omega_n/Q_n$, and we have the expression

$$\dot{E}_{bno} = -\frac{Q_n}{Q_{on}} r_{sn} I_o e^{j\phi_b} \equiv -E_{bno} e^{j\phi_b}, \tag{15'}$$

where E_{bno} is the real amplitude or the peak strength of the induced field. The induced field is out of phase with that of the beam bunch and proportional to the shunt impedance.[3] In the present-designed accelerator, $\phi_b = \frac{\pi}{6}$ and $Q_b \gtrsim 2\, Q_{on}$, so that the approximation of (15') would be correct.

Superposing the induced field to the external field and repeating the process in paragraph 3, we have the following results as,

$$\frac{2\Delta W_n}{\omega_n} = -\frac{r_{sn}}{Q_{on}} \frac{I_o E_{eno} \sin \phi_{bn}}{E_{eno}^2 + E_{bno}^2 - 2E_{eno} E_{bno} \cos \phi_{bn}}$$

$$= -\frac{1}{Q_n} \frac{B_n \sin \phi_{bn}}{1 + B_n^2 - 2B_n \cos \phi_{bn}} \tag{16a}$$

$$\frac{1}{Q_b} = \frac{r_{sn}}{Q_{on}} \frac{I_o (E_{eno} \cos \phi_{bn} - E_{bno})}{E_{eno}{}^2 + E_{bno}{}^2 - 2E_{eno} E_{bno} \cos \phi_{bn}}$$

$$= \frac{1}{Q_n} \frac{B_n \cos \phi_{bn} - B_n{}^2}{1 + B_n{}^2 - 2\, B_n \cos \phi_{bn}} \tag{16b}$$

instead of (10a) and (10b), with the beam loading parameter,

$$B_n = \frac{E_{bno}}{E_{eno}} = \frac{Q_n}{Q_{on}} \frac{r_{sn} I_o}{E_{eno}} , \tag{17}$$

E_{eno} being the n^{th} harmonic amplitude of external field. If we take r_{sn} as 20 $M\Omega/m$, I_o as 0.1 A (the maximum design value), E_{eno} as 4 MeV/m and $Q_n = Q_{on}$, we find $B_n = 0.5$, leading to $\Delta\omega_n/\omega_n = 1.5 \times 10^{-5}$ ($\sim$ 10 kc for an 800 Mc accelerator) and $Q_{bn} = 4 \times 10^4$ for $Q_{on} = 2 \times 10^4$ and $\phi_{bn} = \pi/6$.

5. Self-Consistent Field

Up to the preceding section, the reaction of the field on the charged particles is neglected. In order to find the complete picture, we must solve the self-consistent field taking into account the induced current. The self-consistent field is the solution of the combination of the circuit equation and the equation of motion (electronic equation), this method has been extensively developed in studies of traveling wave tubes.[4] In contrast to the usual TWT theory, the following points should be considered:

a. The injected beam is already tightly bunched and the induced field is important. Also some of the nonlinear behavior should be considered.

b. The motion of the charged particle is relativistic.

c. The circuit is used for standing waves and not for traveling waves.

For the tightly bunched beam, the so-called small-signal theory should not be valid. In a small-signal theory, the velocity, v, and the linear charge density, ρ, of particles are expressed as

$$v = v_o + v_1 e^{j(\omega_b t - k_b z)}$$

$$|v_1| << v_o$$

$$\rho = \rho_o + \rho_1 e^{j(\omega_b t - k_b z)}$$

$$\rho_1 << \rho_o$$

The condition for v will still be valid in the present case, but the condition for ρ is violated since $\rho_1 \sim \rho_o$. Such a case is called a moderately large signal case,[5] and since the beam impedance is still high compared with the circuit impedance, the coupling between the wave and the beam should not be so strong. This is seen through estimation of the dimensionless coupling parameter, C, defined by Pierce,[4] the cube root of the ratio of circuit impedance to the beam impedance. As will soon be shown, C is of the order of 10^{-2} for the present case or of the same order as in the usual TWT. The weak coupling nonlinear theories for the TWT have been presented by Nordsiek and others,[6] and some numerical results are also available. Unfortunately, because of the different working conditions, these numerical results cannot be used for the present case.

On the other hand, by comparison of these numerical results with the results from the simple small-signal theory, Pierce pointed out that, if the beam is overbunched, the effect of nonlinearity can be estimated, at least qualitatively and almost quantitatively, from the results of the small-signal linear theory by setting the amplitude of the varying current equal to $2I_o$.[7] In order to obtain the physical picture, we examine the self-consistent field in this manner.

The circuit equation is again equation (1), and we make some modifications in order to express it in terms of the propagation constants. Then, corresponding to (9),

$$\dot{E}_{no} = - \frac{\omega_n r_{sn}}{2Q_{on} v_{ge}} \frac{\Gamma_n}{\left[(\Gamma_n^2 - \Gamma^2) + \alpha \Gamma_n\right]} \dot{I}_1 \tag{18}$$

where $\dot{E}_{no}$ and $\dot{I}$ are the complex amplitudes of field and current, having time and z-dependence as $\exp(j\omega t - \Gamma z)$. Taking the phase velocities, v_{pn} and v_p, for the case without and with the beam respectively, $\Gamma_n = j\,\omega/v_{pn}$ and $\Gamma = j\,\omega/v_p$ $(\approx j\,\omega_n/v_{pn})$. The effective group velocity, v_{ge}, is defined as

$$-j\, v_{ge} \equiv \frac{\omega - \omega_n}{\Gamma_n - \Gamma}, \tag{19}$$

and the attenuation constant α is

$$\alpha \equiv \frac{\omega_n}{v_{ge} Q_{on}} . \tag{20}$$

Equation (18) has the same form as given by Pierce for TWT theory and the solutions for both forward and backward waves.

The equation of motion for the changed particle is expressed by

$$\dot{i}_1 = \frac{e}{m_\ell} \frac{I_o}{v_o^{\ 2}} \frac{\Gamma_b}{(\Gamma_b - \Gamma)^2} \dot{E}_{no} \tag{21}$$

where $\Gamma_b = jk_b = j\,\omega/v_o$. This is also derived in the similar manner as in TWT theory,[2, 4] provided the relativistic effect is taken into the longitudinal phase motion by taking the longitudinal mass $m_\ell = m_o\,(1 - v_o^2/c^2)^{-3/2}$. To derive this equation the standing wave is divided into forward and backward waves, and is assumed to interact with the beam only through the forward wave due to the phase synchronism condition.*

Combining (18) and (21), the self-consistent field is obtained. Taking $\Gamma_b = \Gamma_n$ (or $v_o = v_{pn}$), the solution for the forward wave is given by

$$\delta^2 \left(\delta + \frac{\alpha}{2} \right) = C^3 k_b^{\ 3} , \tag{22}$$

where $\delta = \Gamma_b - \Gamma$ and the coupling parameter C is given by

$$C^3 = \left(\frac{e}{4m_\ell} \frac{r_{sn} I_o}{v_{ge}\, \omega\, Q_{on}} \right) . \tag{23}$$

---

* The particular situation of the π-mode, which is pointed out by Leiss, can be taken into the calculation of shunt impedance, which will be twice that which was calculated for the forward wave only (J. E. Leiss, Minutes of the Conference on Proton Linear Accelerators at Yale University, October 21-25, 1963, p. 74).

The following values of parameters: $\omega = 2\pi \times 800$ Mc/sec, $r_{sn} = 20$ MΩ/m, $Q_{on} = 2 \times 10^4$, $v_{ge} = 4 \times 10^5$ m/sec, $I_o = 0.1$ A, and the proton kinetic energy = 200 MeV, are taken leading to $C = 1.1 \times 10^{-2}$. The estimate of v_{ge} is difficult especially for a π-mode cavity. As is discussed in the Appendix, the group velocity will vary along the guide to keep the field distribution uniform in an actual cavity. However, assuming the uniform cell structure as is in the present approximation, we may use its average value $<v_g> = \omega L/2\, Q_{on}$. This is about 3.7×10^5 m/sec for L = 3 m.

Now the two limiting cases will be considered; one is $\alpha << Ck_b$ and the other is $\alpha >> Ck_b$. As is shown later, the present case is the latter, while such effects as beam blow-up would occur in the former.

(1) Case 1, $\alpha << Ck_b$

As in the TWT theory, we have three roots of (22) for the forward wave. They are

$$\delta_1 = \frac{1}{2} (\sqrt{3} - j)\, Ck_b \tag{24a}$$

$$\delta_2 = \frac{1}{2} (-\sqrt{3} - j)\, Ck_b \tag{24b}$$

$$\delta_3 = j\, Ck_b \tag{24c}$$

which correspond to the increasing, decreasing, and unattenuated waves, respectively. We have also another solution for the backward wave which is given by

$$\Gamma_4 = -\Gamma_b - \delta_4 \tag{24d}$$

and

$$\delta_4 \doteq \frac{\alpha}{2} - j\frac{C^3 k_b}{4} \quad . \tag{24d'}$$

Since $C \ll 1$, we must take into account α for the δ_4 and may neglect the second term on the right-hand side.

These four waves should be superposed with the proper boundary condition. A finite beam current $I_1(0)$ at the input ($z = 0$) is considered and the standing wave conditions at the input and the output are taken as follows, at $z = 0$ assuming $|\delta_i| \ll k_b$,

$$\dot{E}_1(0) + \dot{E}_2(0) + \dot{E}_3(0) = \dot{E}_4(0) + \dot{E}_e(0) \tag{25a}$$

$$\frac{\dot{E}_1(0)}{\delta_1^2} + \frac{\dot{E}_2(0)}{\delta_2^2} + \frac{\dot{E}_3(0)}{\delta_3^2} = -jK\dot{I}_1(0) \tag{25b}$$

$$\frac{\dot{E}_1(0)}{\delta_1} + \frac{\dot{E}_2(0)}{\delta_2} + \frac{\dot{E}_3(0)}{\delta_3} = 0 \quad , \tag{25c}$$

and at $z = L$,

$$\dot{E}_1(0)e^{\delta_1 L} + \dot{E}_2(0)e^{\delta_2 L} + \dot{E}_3(0)e^{\delta_3 L} = \dot{E}_4(0)e^{(2jk_b + \delta_4)L} \tag{25d}$$

where suffixes n and o for E are omitted for simplicity. The suffix i (= 1, 2, 3 and 4) of $\dot{E}_i$ corresponds to the above four solutions of δ_i and zero in parenthesis means $z = 0$. The constant K is given by

$$K = \frac{m_\ell v_o^2}{e\, k_b I_o} \tag{26}$$

from (21), and the third condition (25c) comes from the assumption of $v_1(0) = 0$. The solutions for $\dot{E}_i$ are given by

$$\dot{E}_i(0) = (-1)^{i+1} \frac{\dot{E}e(0)\,\Delta_{i1} + jK\dot{I}_1(0)\,\Delta_{i2}}{\Delta}, \tag{27}$$

where

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & -1 \\ \frac{1}{\delta_1^2} & \frac{1}{\delta_2^2} & \frac{1}{\delta_3^2} & 0 \\ \frac{1}{\delta_1} & \frac{1}{\delta_2} & \frac{1}{\delta_3} & 0 \\ e^{\delta_1 L} & e^{\delta_2 L} & e^{\delta_3 L} & -e^{(2jk_b + \delta_4)L} \end{vmatrix} \tag{28}$$

and $\Delta_{i,j}$ is the subdeterminant of the i^{th} column and the j^{th} row. Taking δ_4 as $\alpha/2$,

$$\dot{E}_1(0) = \frac{\left(\dot{E}_e(0) - jK\dot{I}_1(0)\right)\delta_1^2\left[1 - e^{-(2jk_b + \frac{\alpha}{2})L}\,\frac{\delta_2 e^{\delta_2 L} - \delta_3 e^{\delta_3 L}}{\delta_2 - \delta_3}\right]}{3 - e^{-(2jk_b + \frac{\alpha}{2})L}\left(e^{\delta_1 L} + e^{\delta_2 L} + e^{\delta_3 L}\right)} \tag{29a}$$

The components $\dot{E}_2(0)$ and $\dot{E}_3(0)$ are found by interchanging subscripts and the component $\dot{E}_4(0)$ is

$$\dot{E}_4(0) = \frac{e^{-(2jk_b + \frac{\alpha}{2})L}\left(\dot{E}_1^0 e^{\delta_1 L} + \dot{E}_2^0 e^{\delta_2 L} + \dot{E}_3^0 e^{\delta_3 L}\right)}{3 - e^{-(2jk_b + \frac{\alpha}{2})L}\left(e^{\delta_1 L} + e^{\delta_2 L} + e^{\delta_3 L}\right)} , \tag{29b}$$

where

$$\dot{E}_i^0 = \dot{E}_e(0) - jK\dot{I}_1(0)\,\delta_i^2 . \tag{30}$$

Terms having the form proportional to $jK\dot{I}_1(0)\delta_i^2$ give the induced field.

(2) Case 2, $\alpha >> Ck_b$

In this case the corresponding four roots of δ are

$$\delta_1 \doteq \frac{1}{\sqrt{2}}(1 - j)\, Dk_b \tag{31a}$$

$$\delta_2 \doteq -\frac{\alpha}{2} \tag{31b}$$

$$\delta_3 \doteq -\frac{1}{\sqrt{2}}(1 - j)\, Dk_b \tag{31c}$$

$$\delta_4 \doteq \frac{\alpha}{2} \tag{31d}$$

assuming $\alpha << k_b$. The modified coupling parameter D is

$$D = \sqrt{\frac{2\ Ck_b}{\alpha}}\ C. \tag{32}$$

General solutions are also obtained from (27) and (28), while, in the first approximation, they are

$$\dot{E}_1(0) \doteq \frac{\frac{\delta_1}{\alpha} e^{\frac{\alpha}{2} L}}{2 \sinh \frac{\alpha}{2} L} \dot{E}_e(0) - \frac{jK \delta_1^{\ 2}}{2} \dot{I}_1(0) \tag{33a}$$

$$\dot{E}_2(0) \doteq \frac{e^{\frac{\alpha}{2} L}}{2 \sinh \frac{\alpha}{2} L} \dot{E}_e(0) + \left(\frac{e^{\frac{\alpha}{2} L} - \cosh \delta_1 L}{\sinh \frac{\alpha L}{2}} \right) \frac{jK \delta_1^{\ 2}}{2} \dot{I}_1(0) \tag{33b}$$

$$\dot{E}_3(0) \doteq - \frac{\frac{\delta_1}{\alpha} e^{\frac{\alpha}{2} L}}{2 \sinh \frac{\alpha}{2} L} \dot{E}_e(0) - \frac{jK\delta_1^{\ 2}}{2} \dot{I}_1(0) \tag{33c}$$

$$\dot{E}_4(0) \doteq \frac{e^{-\frac{\alpha}{2} L}}{2 \sinh \frac{\alpha}{2} L} \dot{E}_e(0) + \left(\frac{e^{-\frac{\alpha}{2} L} - \cosh \delta_1 L}{\sinh \frac{\alpha L}{2}} \right) \frac{jK \delta_1^{\ 2}}{2} \dot{I}_1(0) \ , \tag{33d}$$

where we have assumed the resonance condition $2k_b L = 2\ \pi\ \times$ integer. When the phase shift due to the beam is sufficiently small as is expected in the present design, this assumption will not change the results. In the actual accelerator, α will also vary as z increases, for v_g varies. As we have used the average value of $<v_g>$ $(= \omega_n L/2\, Q_{on})$ for the uniform-cell assumption, one may use the corresponding $\alpha = \omega_n / <v_g> Q_{on} = 2/L$. This value of α is about ten times larger than the parameter Ck_b for L = 3 m so that we can easily see, from numerical checks, that the above expressions (33a) ~ (33d) should be sufficiently close to exact calculated values. Using (20), (23), (26), (31a) and (32), the induced field given by the summation of the second terms in E_i's is found to be expressed as

$$\dot{E}_b = - f(z)\, r_{sn}\, \mathring{I}_1(0) \quad , \tag{34}$$

where f(z) is a function of z taking almost real-positive values. Using the above stated values for α, C and other parameters, f(z) is estimated to be about 0.27 at z = o and 0.73 at z = L. It is interesting to compare these values with that given by (15'), taking $I_1(0) = 2I_o$. Since the coupling effect between the beam and the field was neglected, f(z) was constant and equal to 0.5 in the preceding section. Taking into account the coupling interaction in a lossy guide, the induced field increases as z increases. This is because the assumed interaction is one-directional or the beam interacts only with the forward wave.

6. Discussion of the Beam Blow-up Effect

In a linear accelerator, which is heavily loaded by the coupling interaction between the beam and the wave, the so-called beam blow-up effect would occur as in electron linacs and traveling wave tubes. The beam blow-up effect in electron linacs is caused by the excitation of a deflecting mode HEM having a backward group velocity. Theoretical studies have been done by a method similar to the theory of backward wave tube oscillators leading to good agreement with experiments.[8, 9] Of course, a similar mechanism is possible in high energy proton linacs; however, if we use the π-mode structure, the excitation of the deflecting mode is only possible at its space-harmonics having a forward group velocity. Therefore, the wave excitation of a backward wave tube type cannot occur in this case. The regenerative wave excitation will only occur through the standing wave type interaction, or the feedback due to the backward phase velocity wave, when the loss at the walls has been compensated by the wave amplification due to the beam. Furthermore, the effective shunt impedance to excite such a space-harmonic wave should be much smaller than that for the fundamental wave, therefore one can expect that the starting current for the beam blow-up due to this mode should be relatively high in the present case. This problem has been treated by Gluckstern who obtained a starting current as high as 40 A for the Brookhaven linac.[10]

On the other hand, in a standing wave linac, the regenerative interaction due to feedback through the backward phase velocity wave would also occur at the other modes having high shunt impedances, for example, at fundamental harmonics of TM modes. Similar phenomena have often been observed as an internal feedback oscillation in TWT's with low loss structures and may be referred to as longitudinal beam blow-up.

Since the beam blow-up effect could start even from a small signal such as the noise field, the small-signal theory would be applicable except for nonlinear effects as a coherent interaction due to the tightly bunched beam. The starting current will be sufficiently high to compensate losses, so that the Case 1 of $k_nC >> \alpha$ (still $C < 1$) is probable rather than Case 2. After a numerical calculation in TWT theory including losses,[4] even when $k_nC/\alpha \sim 1$, the solution for δ_i, given by the Case 1 approximation, are still close to the exact calculations.

Then, the starting current for a longitudinal beam blow-up is examined as follows:

First, because of the relatively large value of Ck_bL, simultaneous tuning for the three forward waves cannot be made; i.e. wave 3 will be completely detuned when waves 1 and 2 have been adjusted to resonate the cavity so as to make the wave-beam interaction. Second, the decreasing wave 2 will have high attenuation so that terms of $e^{\delta_2 L}$ can be neglected. Thus, the beam blow-up effect will occur when the denominators in (29a) and (29b) become zero if we consider only the amplification due to wave 1, or

$$3 - e^{\frac{\alpha}{2} L} e^{Re(\delta_1 L)} = 0 \tag{35}$$

Taking $\alpha = \frac{2}{L}$, we have $\sqrt{3}/2\ Ck_bL = 2.10$, and the starting current is given by using (23) as

$$I_{os} = 57.0 \; \frac{m_{\ell} v_o^2 v_{ge} Q_{on}}{e \, r_{sn} \omega \, k_b L^3} \tag{36}$$

$$\doteq 28.5 \; \frac{m_{\ell} v_o^2}{e \, r_{sn} k_b L^2}$$

If we use $v_o = 0.5\,c$, $L = 3\text{m}$, $\omega = 2\pi \times 800$ Mc, and $r_{sn} = 20\ \text{M}\Omega/\text{m}$, we obtain about 5.5 A for the starting current for the TM_{01} mode.* This current is considerably smaller than that estimated for the deflecting mode,[10] though it is sufficiently high compared with the present design. The build-up time, T_b, for this effect is also given by using (19) as

$$T_b = \frac{1}{\mathrm{Re}\, j\,(\omega - \omega_n)} \approx \frac{1}{\mathrm{Re}(v_{ge}\,\delta_1)} \approx \frac{Q_{on}}{\omega} \sim 4\ \mu\text{sec} .$$

Certainly, a similar longitudinal blow-up having a lower starting current may be expected for the excitation of other modes as TM_{01n} $(n > 0)$ rather than TM_{010}. Also, there may be another mechanism of blow-up as a resonant or a coherent interaction suggested by Leiss,[11] which should be a quite nonlinear phenomenon and beyond this study; however, it might be strongly dependent on circuit conditions as discussed by Gluckstern.[10]

At any rate, before such a beam blow-up would appear, one can assume that the actual accelerator should become a decelerator as a TWT amplifier. In the first approximation, the critical current is given by (16b) when the value of Q_b becomes negative. Using (17) and taking the values of parameters used in the paragraph four, one will find about 0.2 A for the critical value. Although this value increases as the field strength due to the external source increases, the actual limiting current of a high energy proton linac will be determined by such a condition.

The author wishes to offer his sincere thanks to Dr. J. P. Blewett for his stimulating interest in this work and for the hospitality of the

---

* At the MURA Linac Conference, we had neglected the effect of the wall loss in (35) and obtained a much smaller value.

Brookhaven National Laboratory. It is a great pleasure to acknowledge the invaluable information about his experimental studies and the helpful discussion of Mr. S. Giordano, who contributed especially to the analysis in the Appendix. The author also thanks Dr. R. L. Gluckstern and Dr. H. Hirakawa for their useful discussions.

APPENDIX

Wave Propagation in a π-Mode Accelerator

As is well known, in the ideal π-mode section, we have essentially zero group velocity from the dispersion curve. On the other hand, in the actual guide with losses, to maintain the field distribution constant along the guide, we need continuous power flow from the external source to the guide, or a finite group velocity in the steady state. Such a condition is only fulfilled with phase shifts and shifts of resonant frequencies which are different for each cell in one section. This has been examined and verified experimentally by Giordano who obtained the dependence of these shifts on cell numbers and Q_o values[12] (Fig. 1 and Fig. 2). We have considered such effects from a simple power relation and a dispersion-equation, giving an idea of the group velocity in the π-mode.*

The energy flow along the guide is denoted by S(z) (J/sec) and the equation of continuity for the energy flow can be written as

$$\frac{\partial U}{\partial t} + \frac{\partial S}{\partial z} + R = 0 \tag{A-1}$$

with the linear density of stored energy U(z) (J/m) and the loss per unit length R(z) (J/sec, m). In the steady state the time derivative is zero so that the first term of (A-1) is eliminated. The energy flow S is expressed quite generally by using the group velocity, v_g, which is also a function of z in the present case, by[13]

$$S(z) = v_g(z)\ U(z) \tag{A-2}$$

Using this expression, (A-1) becomes

$$v_g(z) \frac{\partial U(z)}{\partial z} + \frac{\partial v_g(z)}{\partial z} U(z) = -R(z) \tag{A-3}$$

As is shown in paragraph 2, the peak axial field $E_o(z)$ is related to U(z) through the shunt impedance and the Q-value by means of

---

* A somewhat similar analysis had been done by Nagle and Knapp using an equivalent circuit model (D. E. Nagle and E. A. Knapp, Minutes of the Conference on P. L. A. at Yale University, Oct. 1963, p. 171, E. A. Knapp, LASL Tech Memo P-11/EAK-3, Oct. 1963, unpublished).

$$\frac{E_o^{\,2}(z)}{R(z)} = r_s \qquad \text{and} \qquad \frac{\omega\, U(z)}{R(z)} = Q_o \quad . \tag{A-4}$$

Thus, so far as we consider the nearly uniform structure of cells, the constant field-distribution (i. e. E(z) is constant) corresponds to constant loss and stored energy density (i. e. both R(z) and U(z) are constant). Using the second expression of (A-4), (A-3) becomes

$$\frac{\partial\, v_g(z)}{\partial z} = -\frac{\omega}{Q_o} \quad . \tag{A-5}$$

By integration,

$$v_g(z) = -\frac{\omega}{Q_o}\,(z + \text{const.})\ .$$

If the power is fed from z = o (single feed), it could be assumed to be completely reflected at the end-wall, or $v_g = 0$ at z = L. Then,

$$v_g(z) = \frac{\omega}{Q_o}\,(L - z) \tag{A-6}$$

Now, we assume a simple dispersion equation between the frequency and the propagation constant k like

$$\omega \doteq \omega_o - \frac{\delta\omega}{2}\cos k\, L_o \quad , \tag{A-7}$$

taking the center frequency of the passband, ω_o, the bandwidth, $\delta\omega$, and the unit cell length, L_o. The group velocity is calculated from

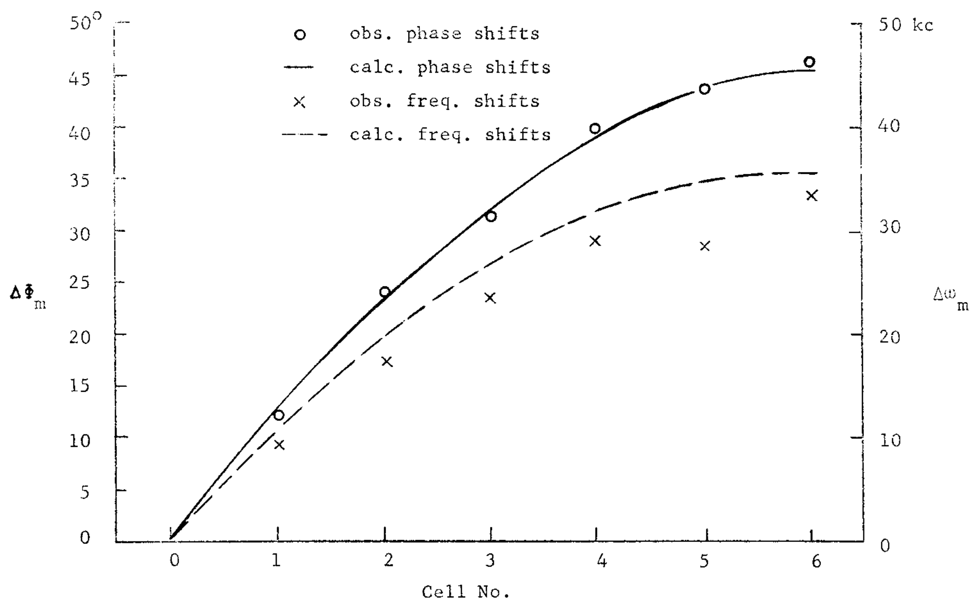

Fig. 1

Dependence of Phase and Frequency Shifts on Cell Numbers

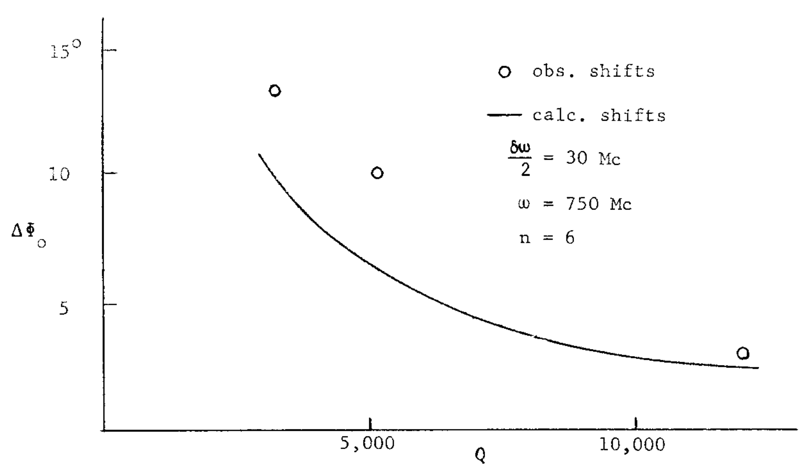

Fig. 2

Dependence of the Total Phase Shift on Q-Value
(Experimental Condition is Different from Fig. 1)

this equation as,

$$v_g = \frac{d\omega}{dk} = \frac{\delta\omega\, L_o}{2} \sin kL_o \tag{A-8}$$

Combining (A-4) and (A-6), the necessary phase shift for the m^{th} cell in a π -mode guide is

$$\Delta \phi_m = \Delta (kL_o)_m = \frac{2v_g}{\delta\omega\, L_o} = \frac{2}{Q_o} \frac{\omega}{\delta\omega} (N-m) , \tag{A-9}$$

where z is replaced by mL_o considering end half-cells (m = 0, 1, 2, ... N for an N-cell section). The phase shifts are accumulated and the total shift is given by

$$\Delta \Phi_o = \sum_{m=o}^{N} \Delta\phi_m = \frac{\omega}{Q_o \delta\omega} N(N+1) \tag{A-10}$$

The observed phase shift is the accumulated phase shift at the center of each cell. Referring to the first cell (m = o), they are

$$\Delta\Phi_m = \Delta\Phi_o - \sum_{n=m}^{N} \Delta\phi_n = \frac{\omega}{Q_o \delta\omega} m(2N - m + 1) . \tag{A-11}$$

Corresponding frequency shifts are also obtained by a Taylor expansion of (A-7) around the π-mode propagation constant, or

$$\omega'_\pi \doteq \omega_\pi + \frac{1}{2} \left(\frac{d^2\omega}{dk^2}\right) (\Delta\phi)^2 . \tag{A-12}$$

Taking $\Delta\omega_m = \omega'_{\pi m} - \omega_\pi$, we have

$$\Delta\omega_m = \frac{1}{4}\delta\omega\ (\Delta\phi_m)^2 = \frac{\omega^2}{Q_{on}^2}\ \frac{1}{\delta\omega}\ (N - m)^2 \quad . \tag{A-13}$$

One can compare these formulae with the experiments. Taking $\omega = 2\pi$ x 880 Mc, $\delta\omega = 2\pi$ x 2.7 Mc, $Q_o = 1.7 \times 10^4$,* and N = 6, the total phase shift and the maximum frequency shift calculated are 46° and 36 kc, respectively. Corresponding experimental values are 47° and 34 kc which are surprisingly close to the calculated values. Also their dependence on cell-numbers gives good agreement between calculations and experiments as shown in Fig. 1, and Fig. 2 gives results of another experiment showing the $1/Q_o$ dependence of the total phase shift, which is also expected by the above analysis, while the calculated absolute values are about 30% smaller than observed, probably because of the very wide band structure.

Now, we have an idea of the group velocity, which varies along the guide. In the present approximation of this article, which is based on a uniform-cell structure, we have no reason to use velocity other than its average value of $\langle v_g \rangle = \omega L/2Q_o$. This velocity also corresponds to the value obtained by the well known dispersion-relation between the phase change and the frequency change of a resonant circuit, in which we consider the whole π-mode section as one cavity.** In a transient phenomenon such as a pulse build-up, one may find another velocity for v_{ge}, which is the velocity of beats between the π-mode and other modes excited by disturbances. The slowest velocity is the velocity of the beat between the fundamental and the next modes, and should be most important. Fortunately, this slowest beat velocity is also of the same order and slightly slower than the above average group velocity in the steady state, in the present designed accelerator.

---

* Observed Q_o-value is 1.5×10^4, whereas a slightly high value is used by taking into account the effect of end-walls.

** The dispersion relation is $\Delta\Psi = -\frac{\Delta\omega}{2\omega_o} Q_o$, and considering

$$\frac{\Delta\lambda_g}{\lambda_g} = \frac{\Delta\Psi}{\pi N}, \quad v_g = \frac{\Delta\omega}{\Delta k} = \frac{\omega_o}{Q}\ \frac{L}{2} \quad .$$

REFERENCES

1. A Proposal for Increasing the Intensity of the Alternating-Gradient Synchrotron at the Brookhaven National Laboratory, BNL 7956, (May 1964).

2. J. C. Slater, Microwave Electronics, (Van Nostrand, 1950), Chap. IV.

3. A. J. Lichtenberg, Rev. Sci. Instr. 33, 395, (1962), J. E. Leiss, NBS Int. Rept. (1958, unpublished).

4. J. R. Pierce, Traveling Wave Tubes, (Van Nostrand, 1950).

5. L. Brillouin, J. App. Phys. 20, 1196 (1949).

6. A. Nordsieck, Proc. IRE 41, 630 (1953), P. K. Tien, et al., Bell System Tech. J. 35, 349 (1956).

7. J. R. Pierce, ibid, Chap. XII.

8. P. B. Wilson, HELPL Rept. No. 297, (June 1963) Stanford University.

9. H. Hirakawa, Jap. J. App. Phys. 3, 27 (1964).

10. R. L. Gluckstern, AADD-38 (July, 1964) Brookhaven National Laboratory Internal Report.

11. J. E. Leiss, ibid and NBS Int. Rept. (Oct. 1962, unpublished).

12. S. Giordano, private communications. The author expresses his thanks for permission to quote here some of the unpublished experimental results.

13. J. C. Slater, ibid, Chap. III and App.