APPLICATION OF CALCULATED FIELDS TO THE STUDY OF PARTICLE DYNAMICS

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We have used a computer program by the name of PARMILA to study certain aspects of the phase and radial motion of particles in linear accelerators. Until recently, we have used expressions for the transit time factor and the radial impulse at the gap, which were derived from the simplest approximation of the field in the gap, that is, a field which is uniform across the geometrical length of the gap, and zero in the drift tube bore.

We have a great deal of information available to us on the actual distribution of fields in the entire linac cell in the form of output from our MESSYMESH program. It is possible to reduce these calculated field distributions to a few coefficients which reflect more precisely the effect of these fields on the particle motion.

I should say that this work was done primarily by Fred Mills and Don Young at a time when I was absent from the laboratory. More recently, I have gone over this work, and I have incorporated it into the PARMILA program.

Let $\vec{\mathcal{E}}\,(r, z, t)$ be the electric field vector at radius r, longitudinal position z, and time t. The fields through which the particles travel, that is the fields near the axis of the linac, can be expressed in terms of the axial component of the electric field on the axis, $\mathcal{E}_z\,(0, z, t)$. At this point we take the time dependence to be sinusoidal and we define $E_z\,(z)$ so that

$$\mathcal{E}_z\,(0, z, t) = E_z\,(z) \cos\,(\omega t + \phi)\,.$$

It is instructive to compare the actual field distribution $E_z\,(z)$ with the simple uniform distributions of field. Figures 1, 2, 3, and 4 facilitate this comparison for typical linac geometries at 50, 100, 150, and 200 MeV. Both curves in each figure are scaled so that

$\frac{1}{L}\int_0^L E_z\,(z)\,dz$ is unity.

Maxwell's equation, in gaussian units, for cylindrical coordinates in a charge-free space, with the further restriction that $B_r = B_z = E_\theta = 0$, yield the following set of nontrivial equations.

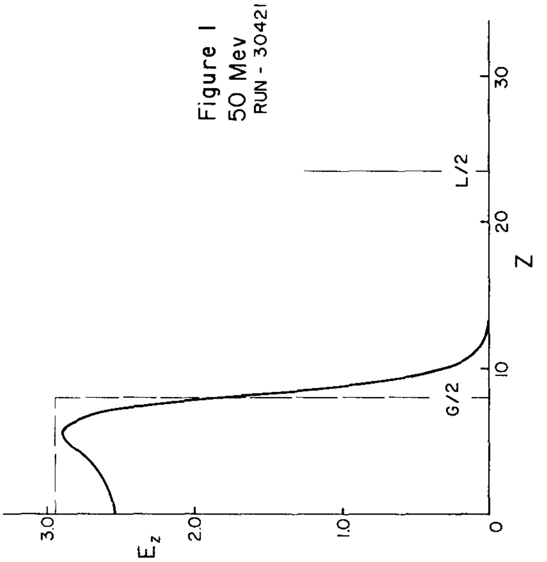

Figure 1
50 Mev
RUN - 30421

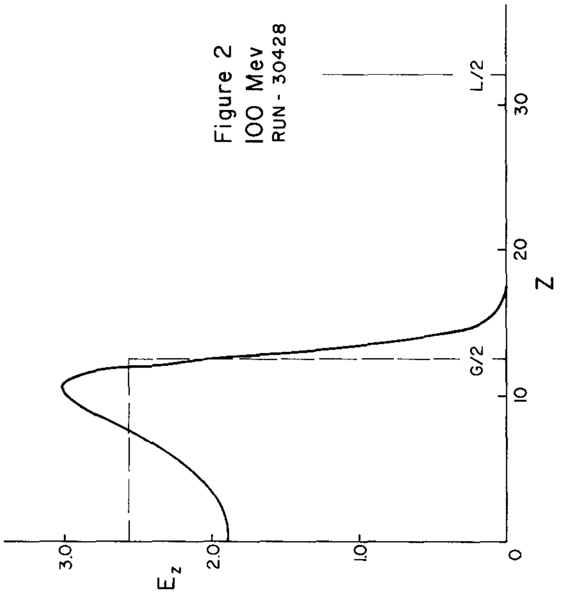

Figure 2
100 Mev
RUN - 30428

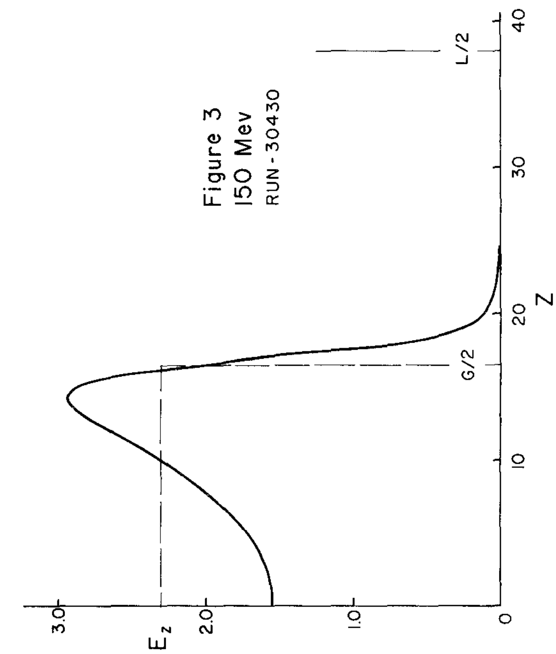

Figure 3
150 Mev
RUN - 30430

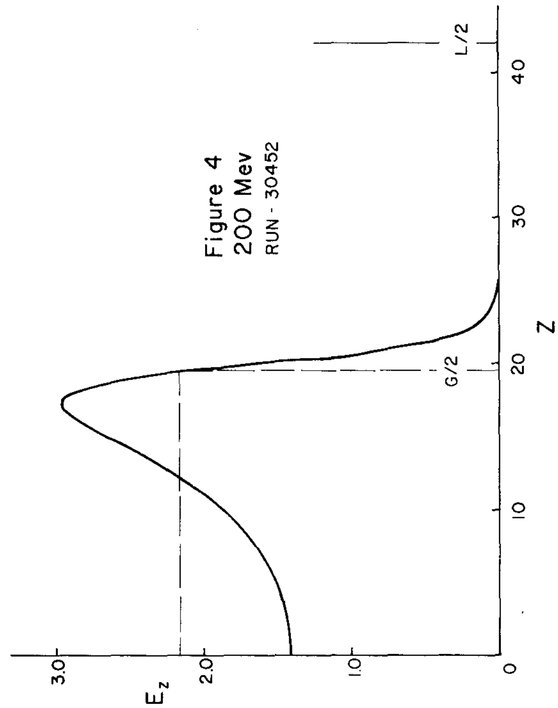

Figure 4
200 Mev
RUN - 30452

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{1}{c}\frac{\partial B_\theta}{\partial t} \tag{1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\ E_r) + \frac{\partial E_z}{\partial z} = 0 \tag{2}$$

$$-\frac{\partial B_\theta}{\partial z} = \frac{1}{c}\frac{\partial E_r}{\partial t} \tag{3}$$

$$\frac{1}{r}\frac{\partial (r\ B_\theta)}{\partial r} = \frac{1}{c}\frac{\partial E_z}{\partial t}\ . \tag{4}$$

We now assume that we know the fields on the axis of the linac [i.e., $\mathcal{E}_z\ (0, z, t)$], and we attempt to get a satisfactory expression for the fields off the axis in terms of $\mathcal{E}_z\ (0, z, t)$. We employ an iterative procedure to get the n^{th} order field in terms of the $n - 1^{st}$ order solution. This procedure yields the following expressions for $\mathcal{E}_z\ (r, z, t)$, $\mathcal{E}_r\ (r, z, t)$ and $B_\theta\ (r, z, t)$.

$$\mathcal{E}_z\ (r, z, t) = \mathcal{E}_z\ (0, z, t) - \frac{r^2}{4}\left(\frac{\partial^2\ \mathcal{E}_z\ (0, z, t)}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\ \mathcal{E}_z\ (0, z, t)}{\partial t^2}\right) \tag{5}$$

$$+ \text{ terms in } r^4 \text{ and higher}$$

$$\mathcal{E}_r\ (r, z, t) = -\frac{r}{2}\frac{\partial \mathcal{E}_z\ (0, z, t)}{\partial z} + \frac{r^3}{16}\frac{\partial}{\partial z}\left(\frac{\partial^2\ \mathcal{E}_z\ (0, z, t)}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\ \mathcal{E}_z\ (0, z, t)}{\partial t^2}\right) \tag{6}$$

$$+ \text{ terms in } r^5 \text{ and higher}$$

$$B_\theta\ (r, z, t) = \frac{r}{2\ c}\frac{\partial\ \mathcal{E}_z\ (0, z, t)}{\partial\ t} - \frac{r^3}{16\ c}\frac{\partial}{\partial t}\left(\frac{\partial^2\ \mathcal{E}_z\ (0, z, t)}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\ \mathcal{E}_z\ (0, z, t)}{\partial t^2}\right) \tag{7}$$

$$+ \text{ terms in } r^5 \text{ and higher}\ .$$

We now proceed to use these fields to evaluate the energy gain and the radial impulse imparted to a particle on traversing a particular linac cell.

The energy gain ΔE is

$$\Delta E = \int_{\text{path}} \vec{\mathcal{E}} \cdot \vec{ds} = \int_{\text{path}} (\mathcal{E}_z \, dz + \mathcal{E}_r \, dr)$$

which can be written, with the fields above, as

$$\Delta E = \mathcal{E}_z(0, z, t)\, dz - \frac{r^2}{4} \int \left(\frac{\partial^2 \mathcal{E}_z(0, z, t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \mathcal{E}_z(0, z, t)}{\partial t^2} \right) dz - \int \frac{r}{2} \frac{\partial \mathcal{E}_z(0, z, t)}{\partial z}\, dr \quad . \qquad (8)$$

Before evaluating these integrals, we make a few definitions. We define the cavity length L and the average axial electric field E_o by

$$\int_{\text{cell}} dz = L$$

and

$$\int E_z(z)\, dz = E_o L \; .$$

We define the origin of z by requiring

$$\int E_z(z) \sin \frac{2\pi z}{L}\, dz = 0 \; .$$

We define the transit time T by the relation

$$\int E_z(z) \cos \frac{2\pi z}{L}\, dz = E_o L T$$

and we define an S factor by

$$\int z\, E_z(z) \sin \frac{2\pi z}{L}\, dz = E_o L^2 S .$$

I will outline the evaluation of the first integral on the right-hand side of Eq. (8) to illuminate the meaning of the terms α and S.

$$\int \mathcal{E}_z (0, z, t)\, dz = \int E_z(z) \cos(\omega t + \phi)\, dz$$

where $t = \frac{z}{V}$. Let $\frac{1}{V} = \frac{1}{V_s}(1 + \alpha)$, where V_s is the velocity of the synchronous particle. Noting that $\omega t = \frac{2\pi z}{L}(1 + \alpha)$, we write

$$\int \mathcal{E}_z (0, z, t)\, dz = \int E_z(z) \cos\left(\frac{2\pi z}{L} + \phi + \frac{2\pi z\alpha}{L}\right) dz$$

$$= \int E_z(z) \left[\cos\left(\frac{2\pi z}{L} + \phi\right) \cos\frac{2\pi z\alpha}{L} - \sin\left(\frac{2\pi z}{L} + \phi\right) \sin\frac{2\pi z\alpha}{L}\right] dz$$

$$= \int E_z(z) \Big[\left(\cos\frac{2\pi z}{L} \cos\phi - \sin\frac{2\pi z}{L} \sin\phi\right) -$$

$$- \frac{2\pi z\alpha}{L}\left(\sin\frac{2\pi z}{L} \cos\phi + \cos\frac{2\pi z}{L} \sin\phi\right)\Big] dz$$

where we have let $\cos\frac{2\pi z\alpha}{L} = 1$ and $\sin\frac{2\pi z\alpha}{L} = \frac{2\pi z\alpha}{L}$.

For a symmetric gap where $E_z(z)$ is an even function of z, two of the four terms in the latter expression integrate to zero, and we are left with

$$\int \mathcal{E}_z (0, z, t)\, dz = E_o L (T - 2\pi\alpha S) \cos\phi .$$

The factor $(T - 2\pi\alpha S)$ can be interpreted as a transit time factor for particles whose velocity is different from the synchronous velocity (i.e., for $\alpha \neq 0$).

When we evaluate the other two integrals on the right-hand side of Eq. (8), the expression for the energy gain for one linac cell is

$$\Delta E = E_o L \left\{ \left(1 + \frac{\pi^2 r^2}{\lambda^2 \beta^2 \gamma^2}\right) (T - 2 \pi \alpha S) \cos \phi - \frac{r r' \pi}{\beta \lambda} T \sin \phi \right\} . \tag{9}$$

To evaluate the radial impulse imparted to the particle on crossing a gap, we write

$$\Delta p_r = \int_{path} |\vec{F}_r| \ dt = \int_{path} e \left(\mathcal{E}_r - \frac{V B_\theta}{c} \right) dt . \tag{10}$$

Using the expression for the fields given in Eqs. (6) and (7), and eliminating third and higher orders in the variable r, we find the expression for $\Delta r'$ to be

$$\Delta r' = - \frac{e \pi}{m_o c^2 \lambda} \frac{E_o L}{\beta^3 \gamma^3} r \sin \phi (T - 2 \pi \alpha S) \tag{11}$$

where $\Delta r' = \dfrac{\Delta p_r}{m_o c \beta \gamma}$.

It is of interest now to compared the factor $T_a - 2 \pi \alpha S_a$ with the transit time function for a uniform field distribution. The analytic expression for the transit time factor for uniform field distribution perturbed by a bore hole of radius "a",

$$T_u = \frac{\sin \frac{\pi G}{\beta \lambda}}{\frac{\pi G}{\beta \lambda}} \frac{I_o \left(\frac{2 \pi r}{\gamma_s L} \right)}{I_o \left(\frac{2 \pi a}{\gamma_s L} \right)} ,$$

can be expressed as

$$(T_u - 2 \pi \alpha S_u) I_o \left(\frac{2 \pi r}{\gamma_s L} \right) \tag{12}$$

where

$$T_u = \frac{\sin \frac{\pi G}{L}}{\frac{\pi G}{L}} \qquad \frac{1}{I_o\left(\frac{2\pi a}{\gamma_s L}\right)}$$

and

$$S_u = \frac{1}{2\pi I_o\left(\frac{2\pi a}{\gamma_s L}\right)} \left(\frac{\sin \frac{\pi G}{L}}{\frac{\pi G}{L}} - \cos \frac{\pi G}{L} \right)$$

and α has the same meaning as above.

First of all, we note that the radial dependence of Eq. (12) is the same as that of the first term in Eq. (9). Secondly, it is of interest to compare the T_a and S_a obtained from the actual field distribution with the T_u and S_u derived from the uniform field distribution. A comparison is given in the Table I for some typical linac geometries ranging in energy from 2 to 200 MeV.

TABLE I

MESSYMESH Run Number	Energy (MeV)	L cm	G cm	A cm	T_a	S_a	T_u	S_u
25003	2.16	10	2.5	1	0.7354	0.0727	0.8198	0.0280
30243	18.21	29	10	1	0.7726	0.0663	0.8066	0.0546
30421	49.80	47	16	1.5	0.7775	0.0655	0.8126	0.0535
30428	97.73	64	25	1.5	0.7101	0.0839	0.7641	0.0682
30430	148.05	76	33	2	0.6262	0.1055	0.7139	0.0811
30452	195.41	84	39	2	0.5636	0.1183	0.6786	0.0902

From the Table I, one can see that the T_a is from 5 to 16% lower than T_u, and that S_a is 20 to 30% higher than S_u with exception of the 2 MeV results in which S_a is 150% higher than S_u. The actual field distributions, shown in Figs. 1 through 4, were obtained from the last four MESSYMESH runs presented in Table I.

I have made some exploratory runs to determine the transverse acceptance and the phase acceptance for an eight-tank 200 MeV linac. A brief description of the linac is given in Table II. It is described in more detail in MURA Technical Note 472 by Young and Austin. The results of

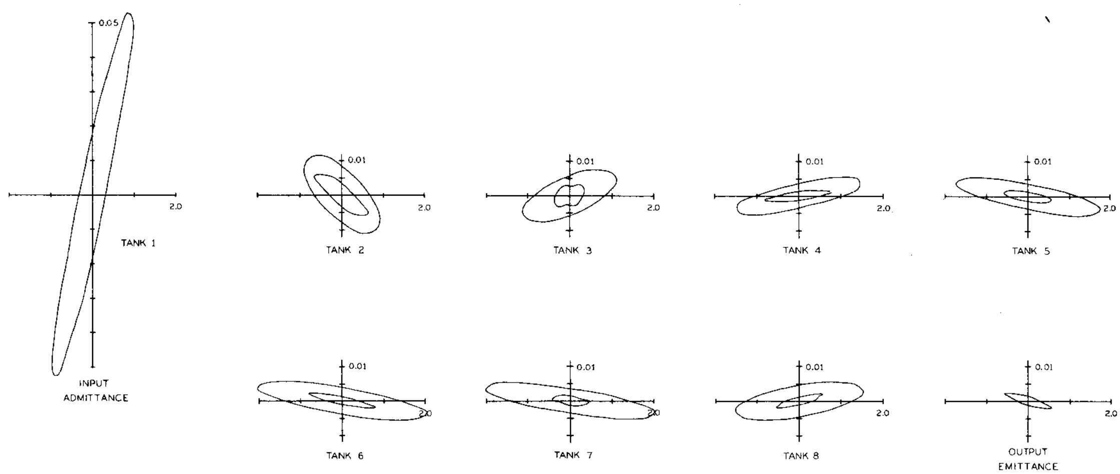

Figure 5 TRANSVERSE PHASE SPACE PLOTS

OUTER CURVE ADMITTANCE OF TANK N

INNER CURVE EMITTANCE OF LINAC THROUGH TANK N-1

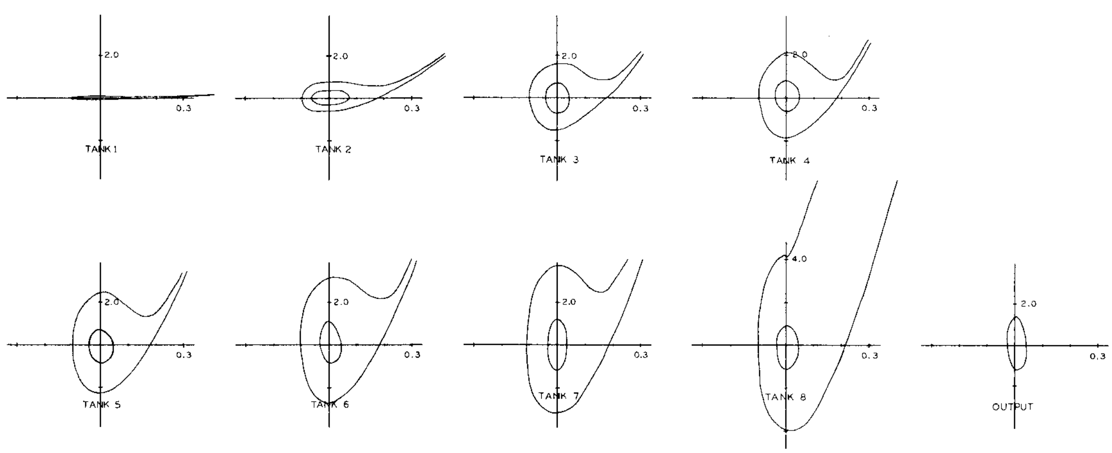

Figure 6 PHASE ACCEPTANCE PLOTS

OUTER CURVE ACCEPTANCE OF TANK N

INNER CURVE PHASE PLOT OF LINAC ACCEPTANCE TRANSFORMED THROUGH TANK N-1 these investigations, which I think are self-explanatory, are shown in Figs. 5, 6, 7, and 8.

TABLE II

Tank No.	1	2	3	4	5	6	7	8
Energy (MeV)	15	46	75	102	128	153	177	200
E_o (MeV/m)	2.0	2.8	2.8	2.7	2.6	2.5	2.4	2.3
Length (m)	9.4	14.8	14.4	15.4	16.3	17.1	17.6	18.5
Power Actual (MW)	2.25	4.95	4.92	4.82	4.96	4.96	4.95	4.79
	Total Length			123.5 m				
	Total Power			36.6 MW				

WALKINSHAW: If you ignore the change in velocity of the particle across a gap and do the analysis as you have done, then you can show that the correct formula is E_o L times that next term in the brackets I_o of some factor of r, then the transit time factor and then cos ϕ . I don't understand where the term in α S comes in unless you are assuming that the particle velocity was changing as it crossed the gap.

SWENSON: The factor $(T - 2\pi\alpha S)$ is effectively a transit time factor for particles of energy E near the synchronous energy E_s. T is the transit time factor for the synchronous particle. The parameter α is defined in the text, but is a function of $(E - E_s)$. α is zero for $E = E_s$.

MILLS: Let me make some comments about this work. About four years ago when we became interested in linacs we began looking mostly for means for computational studies. Our starting point was the report by Panofsky published in 1953. You can see that these formulae are extensions of those in his report.

WALKINSHAW: There is in fact a paradox in some of his formulas.

MILLS: Your specific question about the second term can be answered the following way: In Panofsky's work, only the part of the fields that are traveling with the particle are included. This work includes all the other harmonics in the gap also. About two years ago, Phil Morton began his more complete treatment of the problem which many of you have seen.

WALKINSHAW: I think you will find that Panofsky now would accept that this formula is wrong.

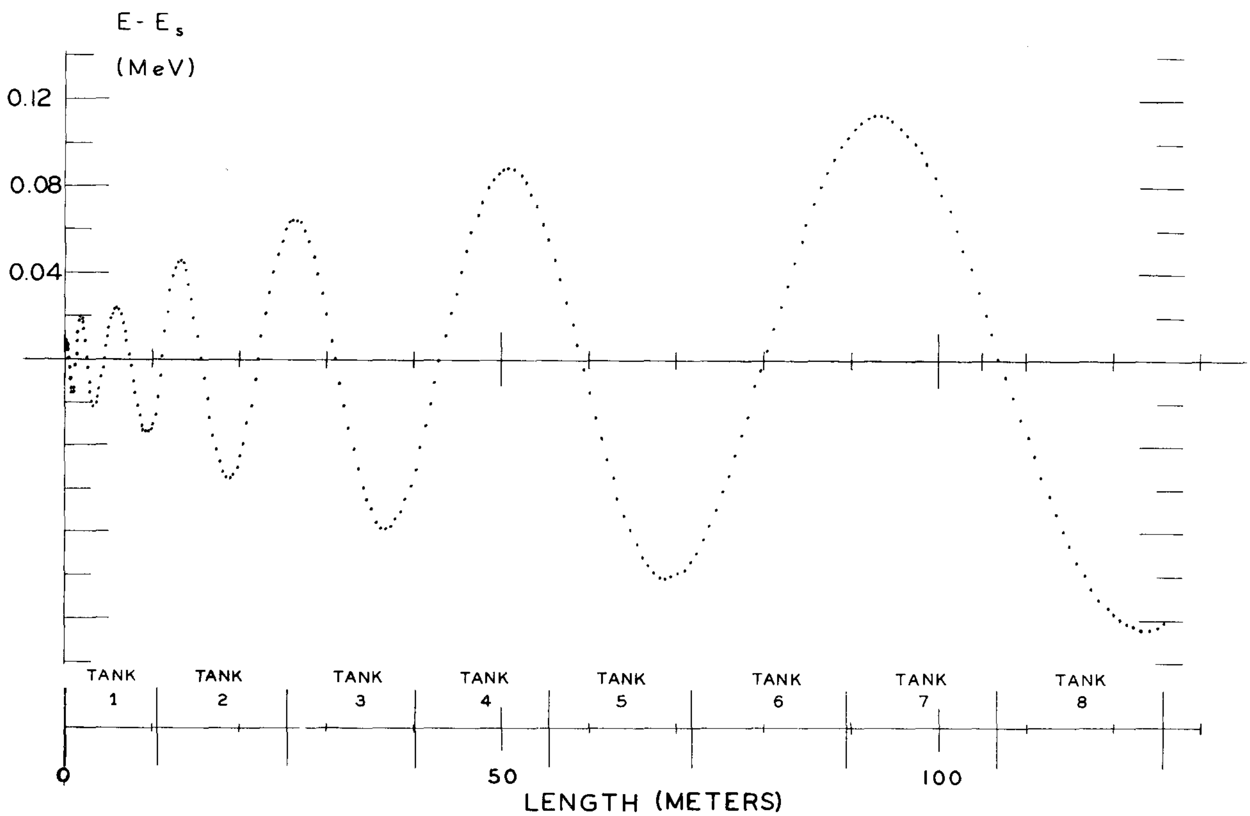

Figure 7 ENERGY OSCILLATION

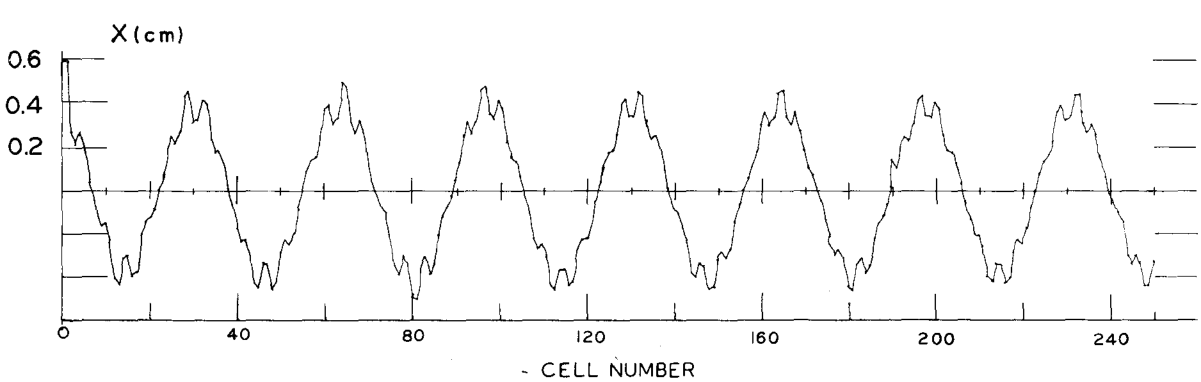

Figure 8 TRANSVERSE OSCILLATION

MILLS: I am not sure which one you mean.

WALKINSHAW: Well, I think that in calculating his transit time factor he put in a term which was dependent on the velocity. This is probably all right when you consider one gap, but when you change this into a differential equation, and consider the phase oscillations of the particle, you find that the phase damping is different from that in a harmonic traveling wave case. This is very curious because you are saying that the harmonic terms are causing some kind of extra phase damping. We wondered once if this was caused by some curious alternating gradient. Then we carried out the gap approximations, and discovered that there are indeed other second-order terms which cancel out the first one, and you come back to the simple approximation.

MILLS: The primary motivation here is the investigation of coupling between the axial and transverse motions. This is only intended to be an expansion in the next leading terms. There is another separate question which I think is related to what you said and that was the following: How adiabatic is the motion? This is a question which was investigated separately by Young and reported in 1961. Indeed the phase motion is not adiabatic in the low energy part of the linac.

WALKINSHAW: I think that the correct answer is the one that you get when you take the harmonic component only. We are quite sure of this. We spent a long time on this and corresponded with Panofsky. He agreed that there was this curious effect. It is quite complicated. What Panofsky was doing here was to integrate across one gap. You have to change this into the differential equation. If you do this by matrix techniques keeping in all higher order terms, including velocity variation, you find the modification to all the terms will cancel exactly the term you have on the board. The explanation that you are tempted to look for is that the higher harmonics, in some curious way, are causing some coherent effect on the linac. This is what started us off; we couldn't really see why this should be so. I think you will find that if you do this properly the harmonic term will in fact give you as accurate an answer as you want.

MILLS: I believe this is done properly in Morton's thesis. I did check to see the nature of the next order term and in fact when one totals up all the harmonics, just those present in this term are there.

OHNUMA: What kind of γ and β do you use? The γ and β are changing continuously across the gap.

SWENSON: We neglect the change of velocity across the gap and use for γ and β some sort of mean value.

OHNUMA: Another question. As I mentioned when I talked, this kind of effect or this kind of calculation might be important when energy is, say, below 50 MeV. But then I seriously doubt, aside from the academic question, in a practical design what the real importance of this kind of calculation would be. This is a point which is not completely clear, because there are all kinds of factors coming into the beam dynamics. Unless the effect is very serious, I do not see the particular importance in an exact point-by-point integration.

GLUCKSTERN: With regard to the point that Bill Walkinshaw made, I think there is another motivation for trying to include some velocity dependence in the formula. I agree with you in what you say in that, if the particle is trapped, then the only thing that can matter is the wave component which is traveling with the velocity of the particle. But an additional quantity of interest, if particles escape longitudinally, is the place where they strike the bore or the irises and cause radioactivity. To answer this question, I think one has to worry about the other wave components. If a particle escapes from a fish, if it is near the border, it will act as if it is stable for quite a way. And not until it gets far enough away so that all the waves average to zero can you take it as not having a change in energy.

WALKINSHAW: Well, I think if you truly analyze your performance, the higher order harmonics are traveling at such a vastly different phase velocity from the particle itself that the effect averages very quickly.

GLUCKSTERN: That is as long as the particle is traveling with the bunch.

WALKINSHAW: Oh, I see. You are saying you may trap them in some of the others.

GLUCKSTERN: No, I was referring to the fundamental wave component only, but for a particle traveling with almost the right velocity. When a particle escapes from a bucket, until it gets to a position where it does not oscillate very much, I think that the effects of the other waves will have to be taken into account.

WALKINSHAW: (Continuation of earlier discussion.) This paradox is a very interesting and amusing one really, because when we saw that the phase damping is different from the harmonic traveling wave case, we started to look for a physical explanation, and you can find one. The reason is this: if you look at the energy gained going across the gap, it will depend on the time it takes across the gap. Now if the particle is making phase oscillations, this means that it is taking different times during its phase oscillation. Part of the time it is going faster across the gap, in which case it gains more energy, and then when it spins through half its oscillation, it gains less and you feel that you have some kind of integrated effect that could cause an increase in the rate of damping. Now that appears to be an explanation, but is the wrong explanation.

SYMON: I do not understand how two different approximations can give two different rates of damping because you can calculate the damping independently of any approximation just from the fact that the area on an energy time plot is rigorously constant independent of any approximations. That means that if the formula gives you a value which disagrees with that, it must be incorrect.

WALKINSHAW: I agree. That is where we started. We got two different rates of damping according to two approximations and then we tried to find out which one is correct and we decided it was the harmonic traveling wave approximation.

SYMON: But you can decide which one is correct by which one gives you the correct area in the end?

WALKINSHAW: Yes, quite so.

FEATHERSTONE: Regarding Dr. Ohnuma's question as to the value of this sort of calculation, I am sort of on the outside here, but looking at the figures over there, for ΔE, the difference between the flat and actual case amounts to more than 10%, which for the person who has to run these things means more than 20% in rf power. I think this is quite significant.

SWENSON: I believe Dr. Ohnuma questioned the significance of the velocity dependent term rather than the term which gives 10% effect which you mentioned. That really results from a better calculation of the transit time factor based on the actual fields in the gap.