ELECTROMAGNETIC FIELD OF TRAVELING CHARGE IN BEAM DUCT

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1. Introduction

The space charge field in a beam duct is expanded in a power series of a parameter α which is proportional to the square root of wall resistivity. The leading term in the expansion is the field in a duct of perfect conductor. The self-defocusing force of the space charge is modified by the induced charge on the wall. Generally, the induced charge gives a longitudinal focusing force, while it invariably is accompanied by transverse defocusing forces (Earnshaw's theorem). Beside these, the whole space charge is attracted toward the wall when the charge distribution has no symmetry around the axis. This contributes to the orbit distortion and the instability of the orbit with respect to transverse displacements.

The second term in the expansion gives a decelerating force for the charge, which makes up for the energy dissipation in the duct wall, together with transverse forces, which are responsible for resistive instabilities.

In the following, Section 2 to 4 deal with the field of traveling charge in a cylindrical duct of perfect conductor. The expansion of the field in the parameter α and the effects of the wall resistivity are discussed in Section 5.

2. Field Equations

The scalar potential ϕ of a space charge field satisfies the wave equation

$$\left(\Delta - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \right) \phi = - \frac{\rho}{\varepsilon_0}. \tag{2.1}$$

Assuming uniform structure of the wall and uniform velocity v of the charge traveling down the duct, we have $\frac{\partial}{\partial t} = -v \frac{\partial}{\partial z}$ and

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\gamma^2} \frac{\partial^2}{\partial z^2} \right) \phi = - \frac{\rho}{\varepsilon_0}$$
$$\frac{1}{\gamma^2} = 1 - \varepsilon_0 \mu_0 v^2 = 1 - \beta^2, \tag{2.2}$$

with the boundary condition for a perfect conducting wall, $\phi = 0$.

The solution of the differential equation (2.2) is given in terms of Green function K

$$\phi(xyz) = \frac{1}{\varepsilon_0} \iiint K(xyz|XYZ)\, \rho(XYZ)\, dX dY dZ, \tag{2.3}$$

with

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\gamma^2} \frac{\partial^2}{\partial z^2} \right) K(xyz|XYZ) = -\delta(x-X)\,\delta(y-Y)\,\delta(z-Z) \tag{2.4}$$

$K(xyz|XYZ) = 0$, for (x,y) on the boundary.

Except for the factor γ, K coincides with the electrostatic Green function in the duct. After Fourier transforms we have

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{h^2}{\gamma^2} \right) \phi_h = - \frac{\rho_h}{\varepsilon_0} \tag{2.5}$$

$$\phi_h(xy) = \frac{1}{\varepsilon_0} \iint K_h(xy|XY)\, \rho_h(XY)\, dX dY, \tag{2.6}$$

and so on, where subscript h means the following:

$$f_h(xy) = \int_{-\infty}^{+\infty} f(xyz)\, e^{-izh}\, dz$$
$$f(xyz) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f_h(xy)\, e^{izh}\, dh. \tag{2.7}$$

For a circular cylinder of radius a,

$$K(r\theta z|R\Theta Z)$$
$$= \frac{\gamma}{2\pi a} \sum_{\ell=1}^{\infty} \frac{J_0(\lambda_{0\ell} \frac{r}{a}) J_0(\lambda_{0\ell} \frac{R}{a})}{\lambda_{0\ell} J_1^2(\lambda_{0\ell})} \exp\left(- \frac{\gamma \lambda_{0\ell} |z-Z|}{a}\right)$$
$$+ \frac{\gamma}{\pi a} \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{J_m(\lambda_{m\ell} \frac{r}{a}) J_m(\lambda_{m\ell} \frac{R}{a})}{\lambda_{m\ell} J_{m+1}^2(\lambda_{m\ell})} \cos m(\theta - \Theta) \exp\left(- \frac{\gamma \lambda_{m\ell} |z-Z|}{a}\right),$$

$$K_h(r\theta|R\Theta) = \frac{1}{2\pi} \left\{ I_0(\tfrac{h}{\gamma} r_1) K_0(\tfrac{|h|}{\gamma} r_2) - \frac{K_0(\frac{|h|}{\gamma} a)}{I_0(\frac{h}{\gamma} a)} I_0(\tfrac{h}{\gamma} r) I_0(\tfrac{h}{\gamma} R) \right\}$$
$$+ \frac{1}{\pi} \sum_{m=1}^{\infty} \left\{ I_m(\tfrac{|h|}{\gamma} r_1) K_m(\tfrac{|h|}{\gamma} r_2) - \frac{K_m(\frac{|h|}{\gamma} a)}{I_m(\frac{|h|}{\gamma} a)} I_m(\tfrac{|h|}{\gamma} r) I_m(\tfrac{|h|}{\gamma} R) \right\} \cos m(\theta - \Theta), \tag{2.8}$$

where $\lambda_{m\ell}$ is the ℓ-th zero of Bessel function $J_m(u)$, and $r_1 = r$, $r_2 = R$ or $r_1 = R$, $r_2 = r$ according to $r < R$ or $R < r$. Profiles of the axially symmetric part of (2.8) are illustrated in Fig. 1. Numerical tables of various Green functions will be published elsewhere.

3. Potential

Using (2.3) potential ϕ is computed for three types of ellipsoidal charge. $\gamma = 1$ is assumed in this section.

i. Uniform density over the region

$$\frac{r^2}{A^2} + \frac{z^2}{B^2} \leq 1. \tag{3.1}$$

ii. Statistical distribution. Uniform density over a hyper-ellipsoid in 6-dimensional phase space

$$\frac{r^2}{A'^2} + \frac{z^2}{B'^2} + \frac{P_r^2}{C^2} + \frac{P_\theta^2}{D^2} + \frac{P_z^2}{E^2} \leq 1$$
$$A' = \sqrt{\tfrac{8}{5}}\, A, \qquad B' = \sqrt{\tfrac{8}{5}}\, B. \tag{3.2}$$

iii. Gaussian distribution

$$\rho = \rho_c \exp\left(-\frac{r^2}{A''^2} - \frac{z^2}{B''^2}\right), \qquad (3.3)$$
$$A'' = \sqrt{\tfrac{2}{5}}A, \quad B'' = \sqrt{\tfrac{2}{5}}B.$$

The total amount of the charge in each ellipsoid is normalized to 10^{10} electron charge = 1.602 × 10^{-9} coulomb. The sizes A', A", B', B", are so determined that the distributions ii and iii have common second moments around the axis as the distribution i . The results of computation are presented in Fig. 2 (equipotentials) and Fig. 3 (potential on the axis). In these figures the potential of the same ellipsoid in free space is shown together. The difference between the duct-field and the free space-field is noticeable particularly in the axial direction.

4. Fields and Forces

The vector potential $\mathbb{A}$ satisfying

$$\left(\Delta - \varepsilon_0\mu_0 \frac{\partial^2}{\partial t^2}\right)\mathbb{A} = -\mu_0 \mathbb{J}$$
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{h^2}{\gamma^2}\right)\mathbb{A}_h = -\mu_0 \mathbb{J}_h \qquad (4.1)$$

is shown to be

$$A_x = A_y = 0, \qquad A_z = \varepsilon_0\mu_0 v\phi.$$

Field $\mathbb{E}$ and $\mathbb{B}$ derived from 4-potential (ϕ ,$\mathbb{A}$)

$$\mathbb{E} = -\operatorname{grad}\phi - \frac{\partial}{\partial t}\mathbb{A}, \qquad \mathbb{B} = \operatorname{rot}\mathbb{A},$$

give Lorentz force on a unit charge

$$f_x = E_x - vB_y = -\frac{1}{\gamma^2}\frac{\partial}{\partial x}\phi$$
$$f_y = E_y + vB_x = -\frac{1}{\gamma^2}\frac{\partial}{\partial y}\phi$$
$$f_z = E_z \qquad = -\frac{1}{\gamma^2}\frac{\partial}{\partial z}\phi \qquad (4.2)$$
$$\mathbb{f} = -\frac{1}{\gamma^2}\operatorname{grad}\phi.$$

Integration of the product $\rho\mathbb{f}$ gives the total force $\mathbb{F}$ on the charge system. Consider a well-defined bunch of charge with a reference point, say the center of gravity, at (x_0, y_0, z_0).

$$\rho(xyzt) = \rho(x-x_0, y-y_0, z-z_0), \quad z_0 = vt$$

The x component of the force $\mathbb{F}$ is

$$F_x = -\frac{1}{\gamma^2}\iiint \rho \frac{\partial}{\partial x}\phi \, dx\,dy\,dz$$
$$= -\frac{1}{2\varepsilon_0\gamma^2}\frac{\partial}{\partial x_0}\int\cdots\int \rho(xyz)\,\mathbf{K}(xyz|XYZ)\,\rho(XYZ)\, dx\,dy\,dz\,dX\,dY\,dZ.$$

Consider Green function in free space

$$\mathbf{F}(xyz|XYZ) = \frac{\gamma}{4\pi}\frac{1}{\{(x-X)^2+(y-Y)^2+(z-Z)^2\}^{\frac{1}{2}}} \qquad (4.3)$$

and a function $\mathbf{G} = \mathbf{K} - \mathbf{F}$ representing the field of the induced charge. The integral

$$\int\cdots\int \rho\,\mathbf{F}\rho\, dx\,dy\,dz\,dX\,dY\,dZ$$

does not depend on the location (x_0, y_0, z_0) of the bunch and does not contribute to the force $\mathbb{F}$. Therefore, with a potential

$$U(x_0 y_0 z_0) = \frac{1}{2\varepsilon_0\gamma^2}\int\cdots\int \rho\,\mathbf{G}\,\rho\, dx\,dy\,dz\,dX\,dY\,dZ$$
$$= \frac{1}{4\pi\varepsilon_0\gamma^2}\int\cdots\int \rho_h^*\,\mathbf{G}_h\,\rho_h\, dx\,dy\,dX\,dY\,dh, \qquad (4.4)$$

one can write

$$\mathbb{F} = -\operatorname{grad} U. \qquad (4.5)$$

Similarly the generalized force F_i for a coordinate x_i of any particular mode of charge deformation is derived from U,

$$F_i = -\frac{\partial}{\partial x_i}U. \qquad (4.6)$$

Note $F_z = -\frac{\partial}{\partial z_0}U = 0$ for a uniform duct.

The function U is given below for a charge filament of effective length 2ℓ and total charge q:

$$\rho(xyzt) = \frac{q}{\pi\ell}\frac{1}{1+\left(\frac{z-z_0}{\ell}\right)^2}\,\delta(x-x_0)\,\delta(y-y_0) \qquad (4.7)$$
$$z_0 = vt$$

$$U(r_0) = -\frac{q^2}{4\pi\varepsilon_0\gamma}\frac{1}{a}\left\{f_0(u) + \frac{1}{2}f_2(u)\left(\frac{r_0}{a}\right)^2 + \cdots\right\}, \quad u = \frac{\gamma\ell}{a} \qquad (4.8)$$

$$f_0(u) = \frac{1}{\pi}\int_0^\infty \frac{K_0(t)}{I_0(t)}e^{-2ut}\,dt, \qquad f_0(0) = 0.4353$$
$$f_2(u) = \frac{1}{\pi}\int_0^\infty \frac{t}{I_0(t)I_1(t)}e^{-2ut}\,dt, \qquad f_2(0) = 1.0027.$$

Functions $f_0(u)$ and $f_2(u)$ are shown in Table I and Fig. 4. Note $f_0(u) \to \log u/(2\pi u)$ and $f_2(u) \to 1/\pi u$ when $u \to \infty$. For infinitely long filament with line density τ, we have

$$U(r_0) = \frac{\tau^2}{2\varepsilon_0\gamma^2}\mathbf{G}(r_0\theta_0|r_0\theta_0)$$
$$= \frac{\tau^2}{4\pi\varepsilon_0\gamma^2}\log\frac{a^2 - r_0^2}{a} \quad \text{per unit length.} \qquad (4.9)$$

Using the force constant $k_{sp} = F_r/r_0 = -\left(\frac{\partial}{\partial r_0}U\right)/r_0$ we have the equation of transverse motion of the charge

$$M\gamma\ddot{r}_0 - k_{sp}r_0 \pm k_{quad}r_0 = 0, \qquad (4.10)$$

where M is the total mass of the filament and $k_{quad} = 2fq\int\frac{\partial}{\partial x}B_y\,dz$ represents the effect of the quadruple magnetic field. Stability of the solution $r_0(t)$ sets the upper bound for k_{sp} .

Consider, for example, the first tank of a proton linac, with drift tube bore a = 1 cm, f = 200 Mc, $\int \frac{\partial}{\partial z} B_y dz$ = 15000 gauss alternating tube by tube. The current limit thus imposed is something around 3 ampere, when u = $\gamma \ell / a$ = 1.

The first term of the series (4.8) represents the effect of the electrostatic force at the entrance of the drift tube. The moving charge obtains kinetic energy -U when entering the bore and gives it back when leaving. With the typical machine parameters above and I = 0.2 ampere, this energy amounts to 0.7 keV per particle.

5. Resistive Forces

Using the time factor $e^{-i\omega t}$ and the complex permeability $\varepsilon' = \varepsilon + i\sigma/\omega$ we write Maxwell equations in a material of finite conductivity σ as following,

$$\mathrm{rot}\ \mathbb{H} = -i\varepsilon'\omega\,\mathbb{E}$$
$$\mathrm{rot}\ \mathbb{E} = i\mu\omega\,\mathbb{H},$$

or

$$(\Delta + \varepsilon'\mu\omega^2)\,\phi = 0$$
$$(\Delta + \varepsilon'\mu\omega^2)\,\mathbb{A} = 0. \qquad (5.1)$$

Fourier transform (2.7) in the coordinate z gives

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{h^2}{\gamma'^2}\right)\phi_h = 0$$
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{h^2}{\gamma'^2}\right)\mathbb{A}_h = 0, \qquad (5.2)$$

where

$$\frac{1}{\gamma'^2} = 1 - \varepsilon'\mu v^2$$
$$= 1 - \varepsilon\mu v^2 - i\,\frac{\sigma\mu v}{h}$$
$$hv = \omega.$$

We need 4-potential (ϕ_h, $\mathbb{A}_h$) satisfying (2.5) and (4.1) in the vacuum and (5.2) in the wall. Consider power series of a parameter $\alpha = \varepsilon_0\gamma/\varepsilon'\gamma'$,

$$\phi_h = \phi_h^{0} + \alpha\,\phi_h^{I} + \alpha^2\phi_h^{II} + \cdots$$
$$\mathbb{A}_h = \mathbb{A}_h^{0} + \alpha\,\mathbb{A}_h^{I} + \alpha^2\mathbb{A}_h^{II} + \cdots. \qquad (5.3)$$

ϕ_h^0, $\mathbb{A}_h^0$ are the potentials obtained in the preceeding sections for a perfect wall duct. $\phi_h^I, \phi_h^{II}, \cdots$ and $\mathbb{A}_h^I, \mathbb{A}_h^{II}, \cdots$ satisfy homogeneous equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{h^2}{\gamma^2}\right)\phi_h^I = 0$$
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{h^2}{\gamma^2}\right)\mathbb{A}_h^I = 0 \qquad (5.4)$$
$$\mathrm{div}\,\mathbb{A}_h^I - i\varepsilon_0\mu_0 hv\,\phi_h^I = 0,$$

and so on. In the range of practical interest, the following is a good approximation for h/γ' and α. We take the liberty of having $\mathrm{Re}\,(h/\gamma') > 0$ and $\mu = \mu_0$.

$$\frac{h}{\gamma'} = \varkappa\,\frac{\beta^{\frac{1}{2}}}{2^{\frac{1}{2}}}\left(\frac{|h|}{\varkappa}\right)^{\frac{1}{2}}(1 \mp i) \qquad \text{for } h \gtrless 0.$$
$$\alpha = \frac{\gamma\beta^{\frac{3}{2}}}{2^{\frac{1}{2}}}\left(\frac{|h|}{\varkappa}\right)^{\frac{1}{2}}(-i \mp 1) \qquad (5.5)$$

$\varkappa$ is the characteristic wave number of the wall material,

$$\varkappa = \sigma\left(\frac{\mu_0}{\varepsilon_0}\right)^{\frac{1}{2}}, \qquad (5.6)$$

with which the skin-depth δ is related as

$$\delta = \left(\frac{2c}{\varkappa\omega}\right)^{\frac{1}{2}}.$$

From the discussions in the preceeding sections, we know $\phi_h^0 = 0$, $A_{zh}^0 = 0$ at the boundary, and $A_{xh}^0 = A_{yh}^0 = 0$ everywhere. The 4-potential in the wall is approximated as following

$$\phi_h'(r,\theta) = \phi_h'(a,\theta)\,e^{-\frac{h}{\gamma'}(r-a)}$$
$$\mathbb{A}_h'(r,\theta) = \mathbb{A}_h'(a,\theta)\,e^{-\frac{h}{\gamma'}(r-a)}. \qquad (5.7)$$

ϕ and $\mathbb{A}$ are continuous across the wall surface, while their normal derivatives satisfy particular boundary conditions. Thus from $H_z = H_z'$, and $H_\theta = H_\theta'$ we have

$$\mathbb{A}_{\theta h}^{I} = 0$$
$$\mathbb{A}_{zh}^{I} = \frac{1}{\gamma h v}\frac{\partial}{\partial r}\phi_h^0 \qquad (5.8)$$

on the boundary. Meanwhile Lorentz condition in the wall gives $\phi_h' = 0$, ensuring

$$\phi_h^I = 0 \qquad (5.9)$$

on the wall surface. Let ψ_h and $\psi_h^\dagger$ be two solutions of the first equation of (5.4), with the boundary consitions

$$\psi_h = -\frac{\partial}{\partial r}\phi_h^0, \qquad \frac{\partial}{\partial r}\psi_h^\dagger = -\frac{1}{a}\frac{\partial^2}{\partial r\partial\theta}\phi_h^0$$

respectively. Then the set of first order potentials satisfying (5.4), (5.8), (5.9) are shown to be

$$\phi_h^I \equiv 0$$
$$\mathbb{A}_{xh}^I \equiv i\frac{\gamma}{h^2 v}\left(\frac{\partial}{\partial x}\psi_h + \frac{\partial}{\partial y}\psi_h^\dagger\right)$$
$$\mathbb{A}_{yh}^I \equiv i\frac{\gamma}{h^2 v}\left(\frac{\partial}{\partial y}\psi_h - \frac{\partial}{\partial x}\psi_h^\dagger\right) \qquad (5.10)$$
$$\mathbb{A}_{zh}^I \equiv -\frac{1}{\gamma h v}\psi_h.$$

Lorentz force on a unit charge in the field of the above potential is

$$f^{I}_{xh} = E^{I}_{xh} - v B^{I}_{yh} = -\frac{1}{\gamma h}\frac{\partial}{\partial x}\psi_h$$
$$f^{I}_{yh} = E^{I}_{yh} + v B^{I}_{xh} = -\frac{1}{\gamma h}\frac{\partial}{\partial y}\psi_h \qquad (5.11)$$
$$f^{I}_{zh} = E^{I}_{zh} = -i\frac{1}{\gamma}\psi_h .$$

The resistive force f^{res} is obtained after Fourier transform of (5.11). It is expressed in terms of a potential Ψ

$$f^{res} = -\frac{1}{\gamma}\,\mathrm{grad}\,\Psi$$
$$\Psi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha}{h}\psi_h e^{izh}\,dh, \qquad (5.12)$$

where ψ_h is given as a Dirichlet's integral

$$\psi_h(xy) = \oint_{r=a}\frac{\partial}{\partial r}\mathbf{K}_h(xy|r\theta)\frac{\partial}{\partial r}\phi^{o}_h(r\theta)\,a\,d\theta \qquad (5.13)$$
$$= \frac{1}{\varepsilon_0}\iint \mathbf{H}_h(xy|XY)\,\rho_h(XY)\,dX\,dY,$$

with

$$\mathbf{H}_h(xy|XY) = \oint_{r=a}\frac{\partial}{\partial r}\mathbf{K}_h(xy|r\theta)\frac{\partial}{\partial r}\mathbf{K}_h(r\theta|XY)\,a\,d\theta. \qquad (5.14)$$

Using (2.8) we have

$$\mathbf{H}_h(r\theta|R\Theta) = \frac{1}{2\pi a}\frac{I_0(\frac{h}{\gamma}r)I_0(\frac{h}{\gamma}R)}{I_0^2(\frac{h}{\gamma}a)} + \frac{1}{\pi a}\sum_{m=1}^{\infty}\frac{I_m(\frac{h}{\gamma}r)I_m(\frac{h}{\gamma}R)}{I_m^2(\frac{h}{\gamma}a)}\cos m(\theta-\Theta) \qquad (5.15)$$

and for h $\to$ 0

$$\mathbf{H}_0(r\theta|R\Theta) = \frac{1}{2\pi a} + \frac{1}{\pi a}\sum_{m=1}^{\infty}\left(\frac{rR}{a^2}\right)^m\cos m(\theta-\Theta). \qquad (5.16)$$

The resistive force acts for a traveling bunch of charge in several ways. To give the general notion f_z^{res} for the charge distribution (4.7) in a circular cylinder is shown below.

$$f_z^{res}(r,z) = -2^{\frac{1}{2}}\beta^{\frac{3}{2}}\frac{q}{4\pi\varepsilon_0 a}\frac{1}{\pi}\times$$
$$\times\int_0^{\infty}\left(\frac{h}{\kappa}\right)^{\frac{1}{2}}\frac{I_0(\frac{h}{\gamma}r)\,e^{-\ell h}}{I_0^2(\frac{h}{\gamma}a)}\{\cos h(z-z_0)+\sin h(z-z_0)\}\,dh \qquad (5.17)$$

For r = 0, we have

$$f_z^{res}(0,z) = -\frac{2^{\frac{1}{2}}\gamma^{\frac{3}{2}}\beta^{\frac{3}{2}}}{(\kappa a)^{\frac{1}{2}}}\frac{q}{4\pi\varepsilon_0 a^2}\{C(u,w)+S(u,w)\} \qquad (5.18)$$

where $u = \gamma\ell/a$, $w = \gamma(z-z_0)/a$. The functions C, S, and C + S are illustrated in Fig. 5. The first terms in { } of (5.18) represents a decelerating force for the charge, while the second term gives a longitudinal focusing force. According to Earnshaw's theorem, the latter should always be accompanied by defocusing forces in transverse direction. As the resistive focusing and defocusing forces are usually much smaller than the corresponding electrostatic forces of previous sections, they need no further consideration. The decelerating force shows up as the result of Ohmic potential drop along the duct due to the wall current. The net decelerating force F_z^{res} is obtained after integrating ρf_z^{res}. The moving charge performs work $v F_z^{res}$ against this force, making up for the energy dissipation in the wall. The net force F_z^{res} for the charge (4.7) is

$$F_z^{res} = -\frac{2^{\frac{1}{2}}\gamma^{\frac{3}{2}}\beta^{\frac{3}{2}}}{(\kappa a)^{\frac{1}{2}}}\frac{q^2}{4\pi\varepsilon_0 a^2}C(u), \qquad u = \gamma\ell/a$$
$$C(u) = \frac{1}{\pi}\int_0^{\infty}\frac{t^{\frac{1}{2}}}{\{I_0(t)\}^2}e^{-2ut}\,dt, \qquad C(0) = 0.3822 \qquad (5.19)$$

Function c(u) is shown in Table I and Fig. 4. Note $c(u) \to 1/(4\sqrt{2\pi}\,u^{\frac{3}{2}})$ when $u \to \infty$. Thus in a stainless steel tube ($\kappa = 4.2 \times 10^8$/meter) of radius a = 1 cm, 10^{12} protons forming a bunch of length 2ℓ = 2 cm receive decelerating field of 0.10 kV/meter when $\gamma \to \infty$.

For a generalized coordinate x_i of charge deformation, the resistive force on the charge system is conveniently derived from a pseud-potential V.

$$F_i^{res} = -\frac{\partial}{\partial x_i}V$$
$$V = \frac{1}{4\pi\varepsilon_0\gamma}\int\cdots\int\frac{\alpha}{h}\rho_h^*\mathbf{H}_h\rho_h\,dx\,dy\,dX\,dY\,dh \qquad (5.20)$$

Using (5.5) we have

$$= \frac{\beta^{\frac{3}{2}}}{2^{\frac{1}{2}}}\frac{1}{4\pi\varepsilon_0}\frac{1}{\kappa^{\frac{1}{2}}}\int\cdots\int\frac{-i\mp 1}{|h|^{\frac{1}{2}}}\rho_h^*\mathbf{H}_h\rho_h\,dx\,dy\,dX\,dY\,dh. \qquad (5.21)$$

Apparently the function V does not exist for a uniform beam with h = 0 component only. Nevertheless the part of V attributed to a perturbation of any wave number can be obtained with this formula. Should the imaginary part of the resistive force can supply enough energy to a particlular mode of oscillation with this wave number, the oscillation builds up with time resulting in an instability of the beam.[1,2]

Acknowledgment

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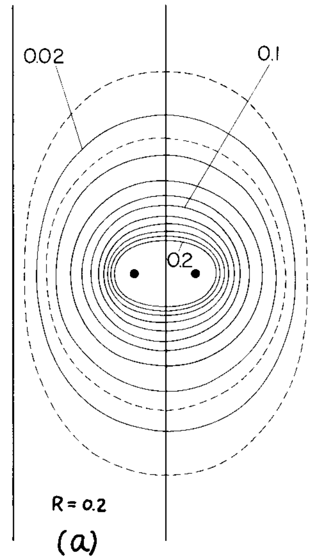
References

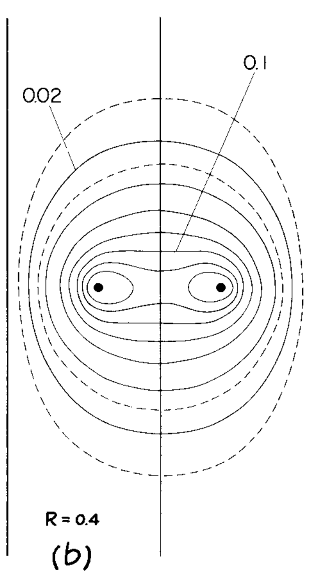
1. V. K. Neil and A. M. Sessler, Rev. Sci. Instr. 36 (1965) 429.

2. L. J. Laslett, V. K. Neil and A. M. Sessler, Rev. Sci. Instr. 36 (1965) 436.

Table I. Functions $f_0(u)$, $f_2(u)$ and $c(u)$.

u	$f_0(u)$	$f_2(u)$	$c(u)$
0.0	0.4353	1.0027	0.3822
0.1	0.3989	0.8225	0.3019
0.2	0.3686	0.6909	0.2440
0.3	0.3430	0.5917	0.2012
0.4	0.3211	0.5149	0.1687
0.6	0.2856	0.4051	0.1235
0.8	0.2579	0.3314	0.9449×10^{-1}
1.0	0.2356	0.2790	0.7479
2.0	0.1677	0.1527	0.3199
3.0	0.1325	0.1040	0.1831
4.0	0.1106	0.7868×10^{-1}	0.1213
6.0	0.8439×10^{-1}	0.5278	0.6700×10^{-2}
8.0	0.6899	0.3967	0.4376
10	0.5873	0.3177	0.3139
20	0.3487	0.1591	0.1114
30	0.2540	0.1061	0.6067×10^{-3}
40	0.2019	0.7957×10^{-2}	0.3941
60	0.1454	0.5305	0.2146
80	0.1148	0.3979	0.1394
100	0.9535×10^{-2}	0.3183	0.9973×10^{-4}





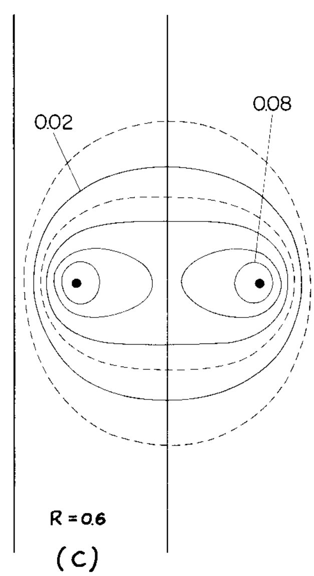


Fig. 1 (a) - (c).
Axially symmetric part of Green function $K(r\theta z | R\Theta Z)$.

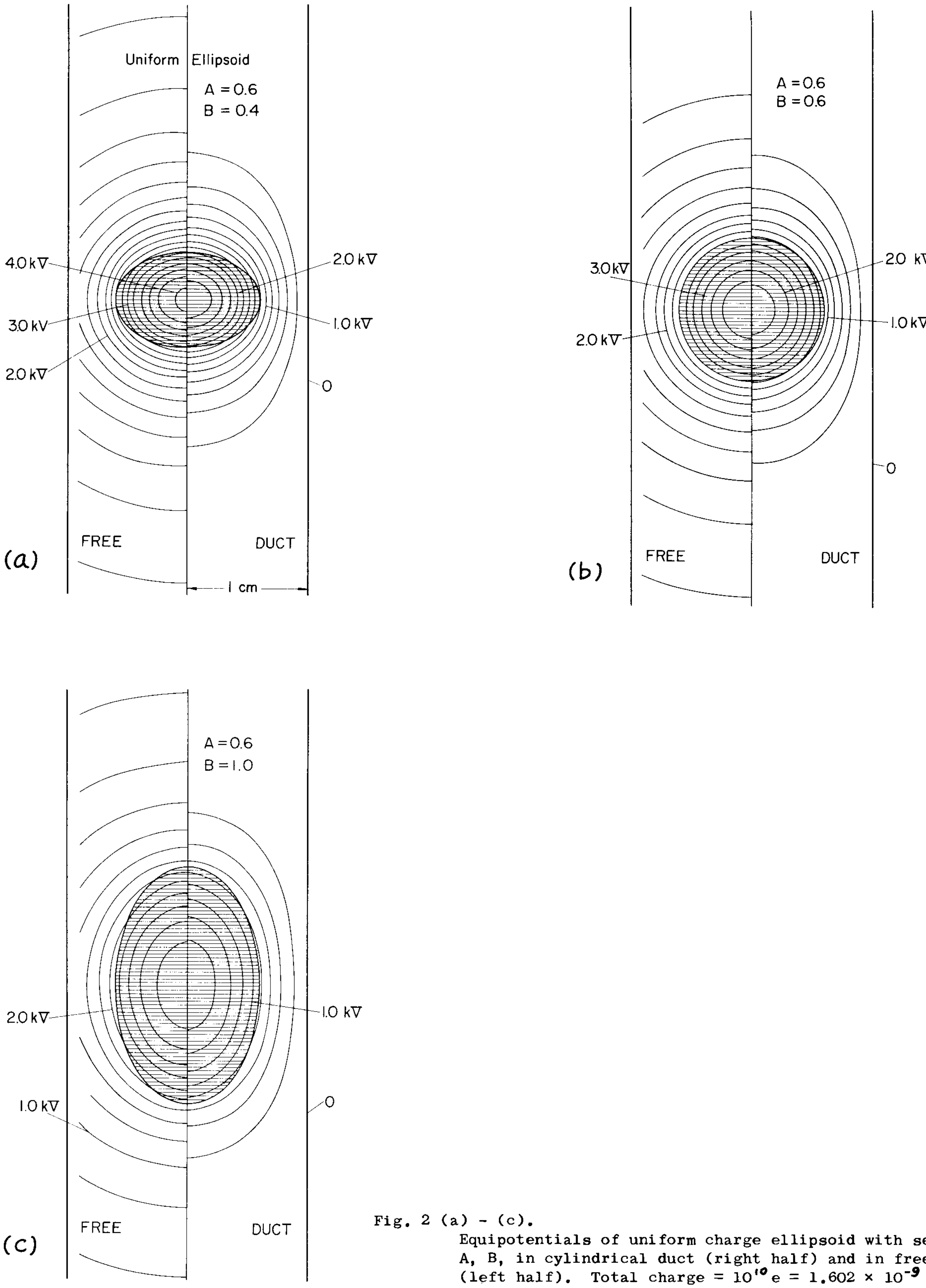


Fig. 2 (a) - (c).
Equipotentials of uniform charge ellipsoid with semi-axis A, B, in cylindrical duct (right half) and in free space (left half). Total charge = 10^{10} e = 1.602 × 10^{-9} coulomb.

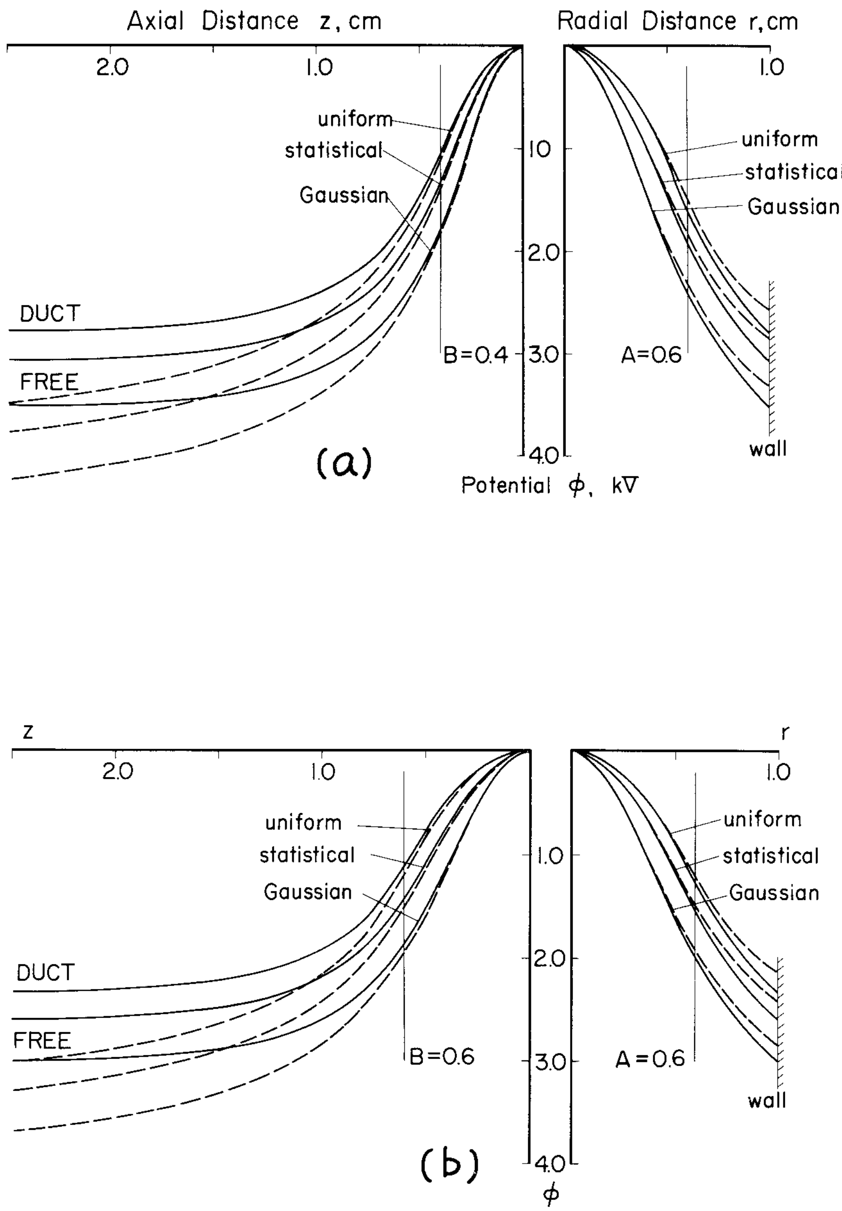
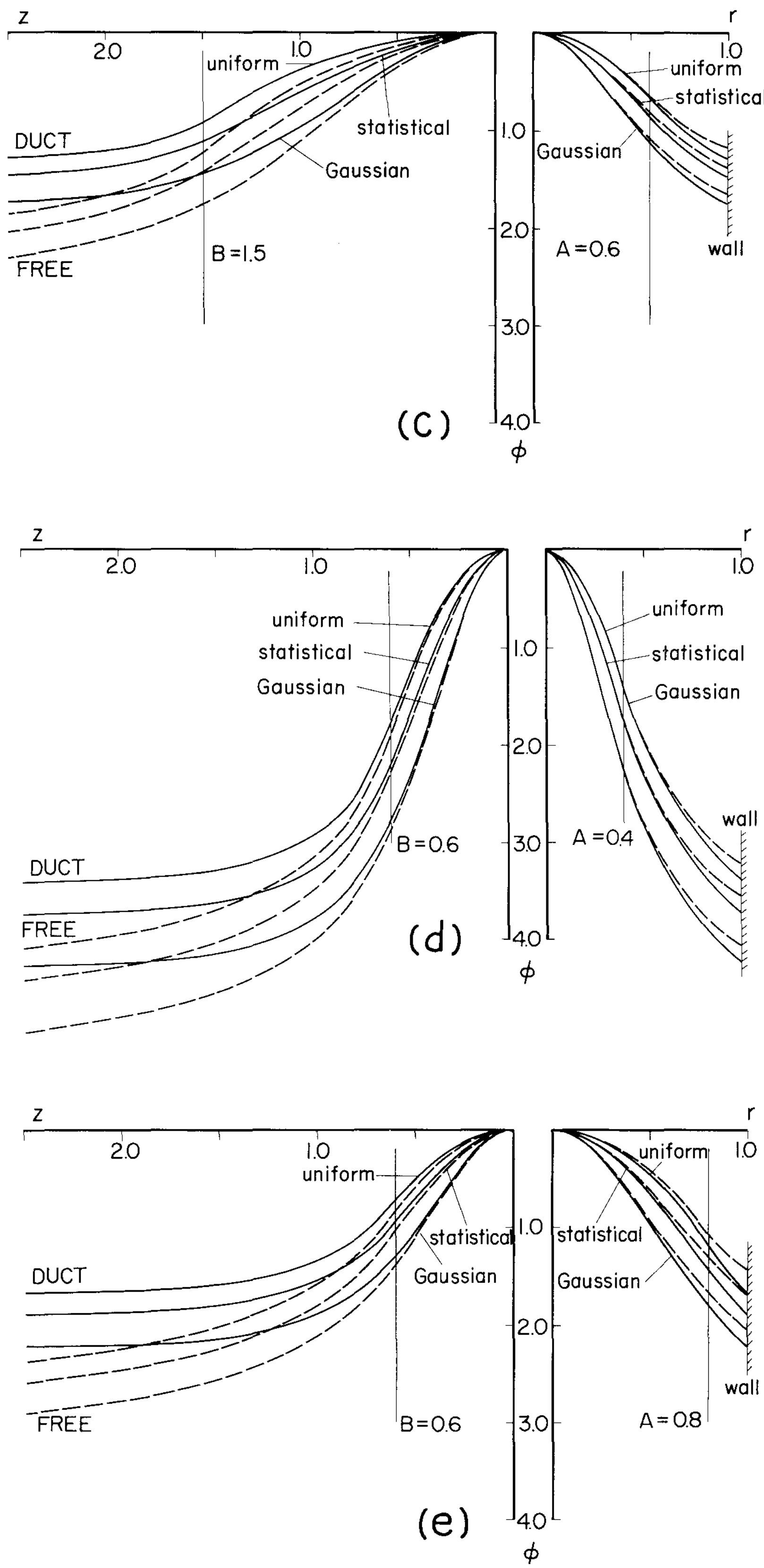


Fig. 3 (a) - (e)
Electrostatic potential on z-axis and r-axis of three types of ellipsoidal charge distribution, (i) uniform, (ii) statistical, and (iii) Gaussian. Total charge = 10^{10} e = 1.602×10^{-9} coulomb. Potentials in cylindrical duct (solid lines) are compared with potentials in free space (dotted lines).


z
2.0
1.0
uniform
statistical
Gaussian
DUCT
FREE
B=1.5
r
1.0
uniform
statistical
Gaussian
A=0.6
wall
1.0
2.0
3.0
4.0
(c)
ϕ
z
2.0
1.0
uniform
statistical
Gaussian
DUCT
FREE
B=0.6
r
1.0
uniform
statistical
Gaussian
A=0.4
wall
1.0
2.0
3.0
4.0
(d)
ϕ
z
2.0
1.0
uniform
statistical
Gaussian
DUCT
FREE
B=0.6
r
1.0
uniform
statistical
Gaussian
A=0.8
wall
1.0
2.0
3.0
4.0
(e)
ϕ

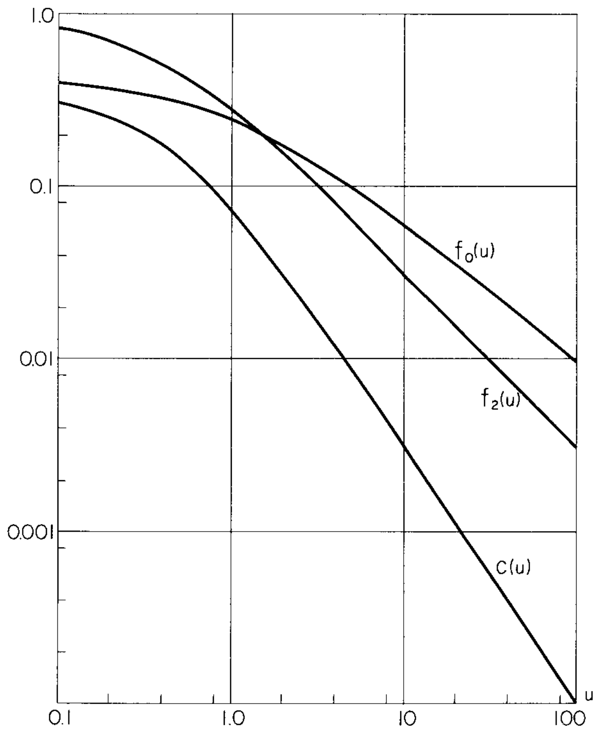


Fig. 4. Functions $f_0(u)$, $f_2(u)$ and c(u).

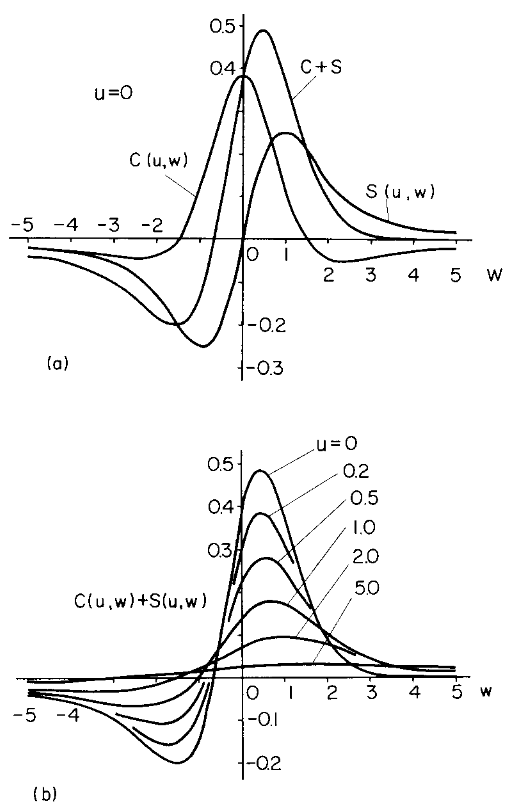


Fig. 5. Functions C(u,w), S(u,w) and C(u,w) + S(u,w).