LONGITUDINAL SPACE-CHARGE EFFECTS IN BUNCHERS

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Calculators of space-charge effects in the early stages of proton linacs have usually made some arbitrary assumption about the distribution of charge density in the bunch. It ought to be possible to make an estimate of the charge density by following the motion of the protons from the injector through the buncher to the accelerator entrance. This talk is concerned with two different attempts to perform such a calculation.

One of us has made a preliminary design for a buncher for our ING project by a method often used in studying klystrons and accelerators. The system under consideration is sketched in Figure 1. Before they cross the first gap the protons are assumed to be uniformly distributed within a cylinder of radius 0.5 cm. A bunch, really a proto-bunch, is defined as the part of the beam which would pass a fixed point in one rf period. In the calculations the motions of three consecutive bunches are followed, one guarding bunch on each side of the bunch under study. Each bunch is divided into a finite number - 20 to 40 - of equal pill-boxes. Each pill-box is then assumed to move as a rigid body, but pill-boxes are allowed to pass through each other. When the force on a particular pill-box is being calculated its charge is assumed to be concentrated at its centre; the charges in all the other pill-boxes are left distributed. The equations of motion of the pill-boxes are integrated simultaneously by a simple numerical method. Although some allowance can be made for the transverse motion of the protons, such effects are not included in the calculations presented here; the motion is assumed to be strictly one-dimensional.

We present the results of the calculation in the form of a graph showing the final energies and positions (with respect to the bunch centre) of particles which were initially at the centres of the pill-boxes. For the sake of comparison Figure 2 shows results with no space charge forces. The isolated crosses in Figure 3 give the results with space-charge forces calculated by the method of pill-boxes. It is apparent that the space-charge forces have a large effect on the action of the buncher.

The pill-box method is a model which replaces something like 10^9 particles by 20 to 40 quasi-particles. Can such a model give meaningful results? One way to find out is to make comparisons with a model which goes to the opposite extreme - an infinity of particles. Let us send the particle charge, e, and its mass, m, to zero keeping the ratio e/m and the charge density fixed. The one-dimensional equation of motion for a test particle is then

$$\frac{d^2Z}{dt^2} = \frac{e}{m}\int \rho(Z',t)K(Z,Z')dZ'; \qquad (1)$$

Z is particle position measured along the accelerator axis and $\rho(Z',t)$ is the linear charge density at position Z' at time t. The form of the kernel K(Z,Z') appropriate to a one-dimensional beam in empty space is

$$K(Z,Z') = \frac{(Z-Z')}{2\varepsilon_o}\left[\frac{1}{|Z-Z'|} - \frac{1}{\sqrt{(Z-Z')^2+a^2}}\right] \qquad (2)$$

MKS units are used and ε_o is the permittivity of empty space; a is the radius of the beam. The function K(Z,Z') jumps discontinuously when Z passes through Z', but remains finite.

In integrating (1) we need to know the initial conditions. Let ζ be the initial value of Z for a particular test particle. The values of ζ lie in a finite interval, $a \leq \zeta \leq b$, corresponding to the set of three bunches under consideration. We shall assume that all test particles have the same initial velocity, v; this assumption is expected to be a very good approximation. It is convenient to consider the Z defined by (1) and the initial conditions just specified as a function of ζ and t, say $Z = f(\zeta,t)$. Then, because $\rho(Z',t)$ can be expressed in terms of $f(\zeta,t)$, (1) may be seen to be an integro-differential equation for the function $f(\zeta,t)$.

Next we come to the important question of how $\rho(Z,t)$ is to be calculated. We shall assume that $f(\zeta,t)$ is a continuous function of ζ for fixed t. The amount of charge between Z_1 and $Z_1+\Delta Z$ at time t is (see Fig. 4a) the same as the amount of charge which lay initially between $Z = \zeta$ and $Z = \zeta+\Delta\zeta$. If the initial charge density was uniform and equal to ρ_o, then

$$\rho(Z_1) = \rho_o\left[\left|\frac{dZ}{d\zeta}\right|_{\zeta=\zeta_1}\right]^{-1} \qquad (3)$$

for $\Delta Z \to 0$. When the function $f(\zeta,t)$ is not monotonic in ζ we can have a situation such as that illustrated in Fig. 4b; then

$$\rho(Z) = \rho_o \sum_j \left[\left| \frac{dZ}{d\zeta} \right|_{\zeta=\zeta_j} \right]^{-1} \qquad (4)$$

The number of terms in the sum is one greater than the number of extrema. It should be noted that the presence of extrema in the curve of $f(\zeta,t)$ versus ζ corresponds to an overlapping of particles in the bunch; some particles have overtaken since other particles which were initially ahead of them.

Note that when $dZ/d\zeta$ is zero the charge density is infinite. The positions of the singularities when $f(\zeta,t)$ has two extrema are sketched in Fig. 5. The singularities are integrable, usually of the form $\rho \sim C[|Z_i - Z|]^{-\frac{1}{2}}$.

The integration of (1) has now become a problem of considerable complexity. In addition to the discontinuities in $K(Z,Z')$ we must also deal with possible singularities in $\rho(Z',t)$. However, there is an easy way to eliminate the second difficulty from the evaluation of the force in (1). Transform the variable of integration from Z' to ζ' by $Z' = f(\zeta',t)$. Each value of Z' can correspond to more than one value of ζ', but the different terms in (4) are exactly what is required to give

$$\frac{d^2Z}{dt^2} = \frac{e}{m} \rho_o \int_a^b K(Z, f(\zeta',t)) d\zeta'. \qquad (5)$$

The form of the integrand in (5) is illustrated in Fig. 6 for the $f(\zeta,t)$ of Fig. 4b with $Z_1 < Z < Z_2$. At a finite number of points the integrand jumps discontinuously as if it were suddenly multiplied by -1. All the extremities in Fig. 6 are at the same distance from the axis; the sharpness of the peak or valley near an extremity depends on the local value of $df/d\zeta$.

The transformation leading to (5) may easily be generalized to the three-dimensional problem. All that is required is that there be a unique velocity for each point of the beam at $t = 0$. It is convenient, though not necessary, to assume that the initial velocities are all equal and that the initial charge density is uniform.

A program has been devised for evaluating the right side in (5). Several integration methods are used, and the values of ζ at which the integrand is to be evaluated are chosen as required for each value of t. It is assumed that $f(\zeta,t)$ for fixed t is a continuous function of ζ with continuous first and second derivatives. The only interpolation required is for values of the relatively smooth function $f(\zeta,t)$, which is assumed to be given at fixed equally spaced values of ζ. This is much superior to any method which requires one to interpolate for the integrand directly. The table of values of $f(\zeta,t)$ is obtained by applying Runge-Kutta integration to (5) for a finite number of fixed values of ζ.

The joined points in Fig. 3 represent the results for the buncher of Fig. 1 calculated by the method of Eq. (5). Computations with double the number of points in the table giving $f(\zeta,t)$ or with double the time-step in the integration of the differential equations give almost identical results. The agreement between the results for the two different methods of calculation shown in Fig. 3 is sufficient to suggest that either is adequate for buncher design. Decreasing the time-step by a factor 4 in calculations by the pill-box method for similar bunchers changed the result by only a small amount.

Before drawing two optimistic a conclusion from the agreement shown in Fig. 3 one should note that no singularities in the charge density have developed by the time the bunch reaches the accelerator. In the curves of energy versus position the singularities correspond to points where the tangent is vertical. Singularities will certainly develop as the bunch proceeds through the accelerator. To provide a better comparison between the two models we have also done calculations for a buncher which, when studied by the pill-box method, did exhibit overlapping. Calculations by both methods are shown in Figs. 7 and 8 for 20 and 40 test/quasi particles per bunch respectively. The two calculations by the continuous-fluid model agree very well with each other; the two by the pill-box method do not. Neither pill-box calculation agrees with the continuous-fluid calculations. Both pill-box calculations have much more structure near the bunch centre than do the continuous-fluid calculations.

At higher beam currents the continuous-fluid method, as now used, also becomes unstable. The calculation of the space-charge force still appears to be satisfactory. The trouble is apparently in the integration of the equations of motion. These equations are effectively differential equations of the form

$$\frac{d^2Z}{dt^2} = F(Z,t). \qquad (6)$$

When $\rho(Z',t)$ is singular the electric field strength, and hence $F(Z,t)$, has a singular first partial derivative with respect to Z. This property invalidates the Runge-Kutta method of integration which has been used. An attempt is being made to devise a better method of integration, one that varies the time-step from particle to particle depending on proximity to a singularity.

DISCUSSION

(G. E. Lee-Whiting)

WARNER, CERN: What was the size of the time-step used in the numerical integration of the equations of motion for the pill-box model? Did it correspond to a step of size $\beta\lambda$ in distance?

LEE-WHITING, AECL: The normal time-step was 4 nanoseconds.

WARNER, CERN: Yes, that certainly corresponds to a distance-step of about $\beta\lambda$.

LEE-WHITING, AECL: But some runs were done with a step of 1 nanosecond.

WARNER, CERN: Do you think that the fine structure in the Z-energy plane could be caused by too large a step in time?

LEE-WHITING, AECL: The structure depends upon the size of the time-step, but I do not believe that it will disappear as the step goes to zero. The results converge slowly as the time-step is decreased. Calculations performed with decreasing time-steps do not indicate that the pill-box results approach the continuous-fluid results. I expect that this would happen only for a much larger number of particles. I can not make my answer as direct as I would like to, because the tests with different time-steps were carried out for a slightly different buncher.

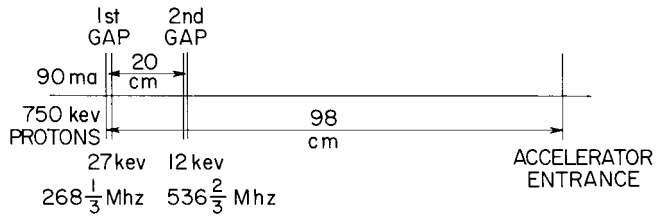

Fig. 1 Sketch of the lay-out of a buncher proposed for the ING project.

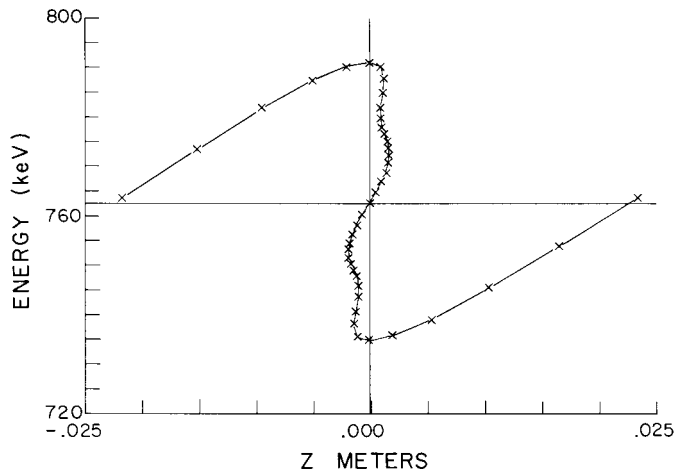

Fig. 2 Plot of energy versus distance from the bunch centre at the instant the bunch centre enters the accelerator. Here space-charge forces are ignored. Before entering the buncher the particles were evenly spaced in Z and all had the same energy.

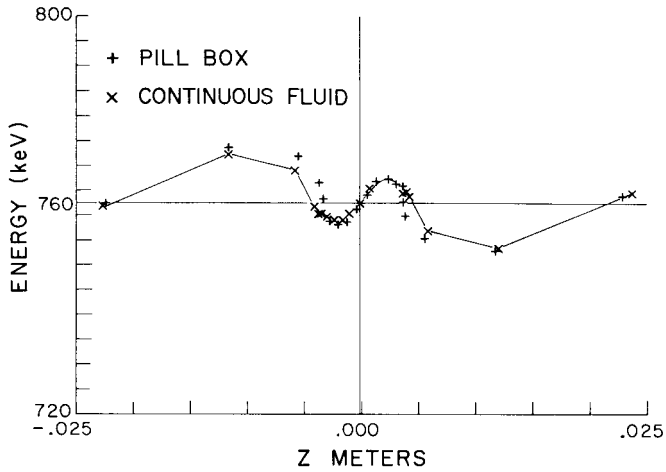

Fig. 3 Plots as in Fig. 2, with space-charge forces included. The isolated crosses were obtained by the pill-box model, the joined points by the continuous-fluid model.

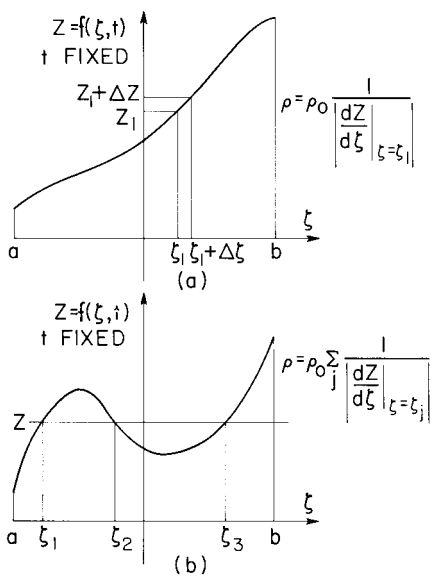

Fig. 4 Graphs to illustrate how the charge density is obtained from the function $f(\zeta,t)$, when it is monotonic (a) and when it has extrema (b).

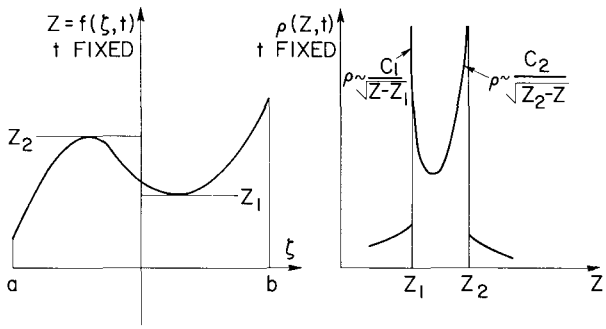

Fig. 5 Illustration of how the singularities in the charge density are related to the extrema of the function $f(\zeta,t)$.

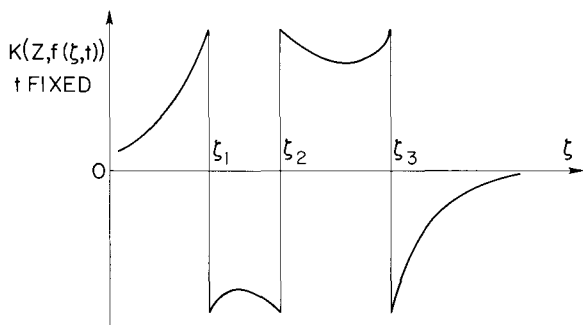

Fig. 6 A sketch of the form of the integrand in Eq. (5) when $f(\zeta,t)$ has the form shown in Fig. 5 and Z lies between Z_1 and Z_2.

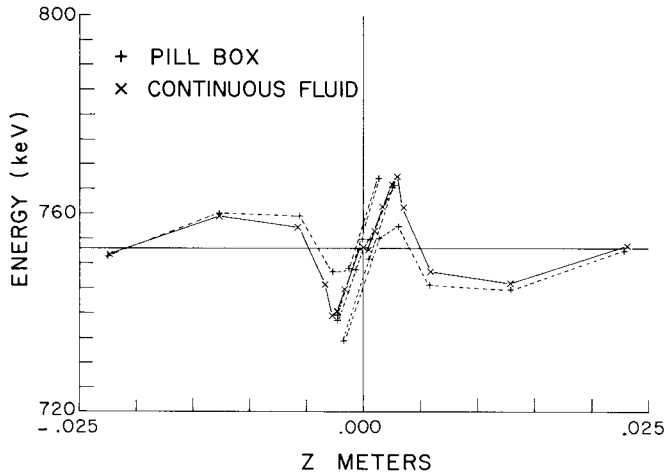

Fig. 7 Plots of energy versus Z at the accelerator entrance for a buncher with energy gains of 27 kev and 10 kev at coincident first and second harmonic gaps; the distance from the gaps to the accelerator entrance is 97.86 cm. The points joined by the solid lines were calculated by the continuous-fluid method, those by the dashed lines by the pill-box method. In each case 20 test/quasi particles were used per bunch.

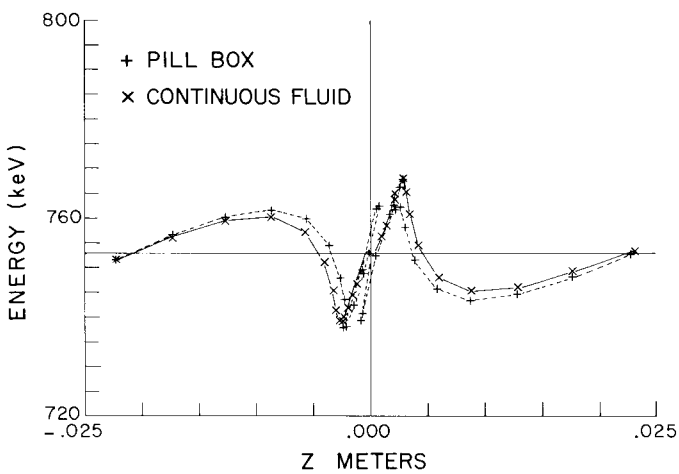

Fig. 8 Same as Fig. 7 for 40 test/quasi particles per bunch. A few of the central quasi particles for the pill-box calculation are not shown.