PHASE SPACE ACCEPTANCE OF A QUADRUPOLE DOUBLET ANALYTIC EXPRESSIONS AND NUMERICAL RESULTS

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Consider a quadrupole of length s and aperture 2R. Let G be the magnetic gradient, p the momentum of the particles forming the beam, and q their charge. We put as usual

$$k = \sqrt{Gq/p} \quad , \qquad \theta = ks$$

In the focusing plane of the quadrupole the phase acceptance is determined by the conditions:

$$-R \leq X_o \leq R$$

$$-R \leq X \leq R$$

and

$$-k X_o \sin \phi + X_o' \cos \phi = 0$$

$$-R \leq X_o \cos \phi + \frac{X_o'}{k} \sin \phi \leq R$$

where X_o, X_o' and X, X' are the phase space coordinates corresponding respectively to the entrance and the exit of the quadrupole and ϕ varies from 0 to θ.

Figure 1 shows the phase space contour corresponding to the focusing plane at the entrance of the quadrupole. The area enclosed is:

$$A_c = kR^2 \left(\theta + 2 \cot \frac{\theta}{2}\right)$$

By a similar method one can obtain the phase space contour corresponding to the defocusing plane. Its shape at the entrance of the quadrupole is given in Fig. 2; the area enclosed is:

$$A_d = \frac{4kR^2}{\sinh \theta}$$

To calculate the phase space acceptance corresponding to the "cd plane"* of a doublet, we consider the area common to the focusing contour, taken at the exit of the focusing element and the defocusing contour, transformed to the exit of the focusing element (Fig. 3). In the same way, to determine the acceptance area corresponding to the "dc plane",* we consider the area common to the focusing contour, taken at the entrance of the focusing element and the defocusing contour, transformed to the entrance of the focusing element.

*Note: The "cd plane" is the plane in which the first quadrupole is convergent and the second is divergent. The "dc plane" is the plane in which the first quadrupole is divergent and the second is convergent.

We restrict ourselves here to the case where the two quadrupoles of the doublet have identical characteristics, i.e. $s_1 = s_2 = s$, $k_1 = k_2 = k$ and therefore $\theta_1 = \theta_2 = \theta$. The acceptance areas corresponding to the "cd plane" and "dc plane" are then equal, i.e., $A_{cd} = A_{dc} = A$ and the situation in the dc plane can be described by a diagram which one would obtain from Fig. 3 by means of a mirror reflection with respect to the Y axis.

The positions of the points of intersection of the focusing and defocusing contours (Fig. 3) depend on the value of the parameter θ. In particular, if the conditions

$$\sin\theta \sinh\theta < 1 + \cos\theta \cosh\theta < 2 + \sin\theta \sinh\theta$$

are satisfied, the intersection point T will always be on the elliptical arc of the focusing contour, whereas the intersection point S will be either on the elliptical arc or on the straight line segment, according to the length of the drift space L between the quadrupoles of the doublet. Putting $kL = \lambda$ the quantity

$$\lambda_S = \cot \theta - \tanh \theta + \frac{1}{\tanh \theta \cosh \theta}$$

defines a separatrix between two modes of operation.

i) For $\lambda > \lambda_S$, the phase acceptance is given by

$$A = kR^2 \left[\frac{2\sqrt{\alpha^2 + \sinh^2\theta}}{\alpha^2 + \cosh^2\theta} + \arc\sin \frac{2\sqrt{\alpha^2 + \sinh^2\theta}}{\alpha^2 + \cosh^2\theta} \right]$$

where:

$$\alpha = \sinh \theta + \lambda \cosh \theta$$

ii) For $\lambda \leq \lambda_S$, the phase acceptance is given by:

$$A = kR^2 \left[(\theta - \tan \theta) - \left(\Psi - \frac{\sin \Psi}{\cosh \theta}\right) + \frac{\left(\frac{1}{\cos \theta} + \frac{1}{\cosh \theta}\right)^2}{\tan \theta + \tanh \theta + \lambda} \right]$$

where:

$$\sin \Psi = \frac{\sinh^2 \theta}{\alpha + \sqrt{\alpha^2 + \sinh^2\theta}\,\cosh\theta}$$

Figure 4 shows the normalized acceptance A/kR^2 as a function of the normalized drift length λ for various values of the parameter θ.

The inequalities:

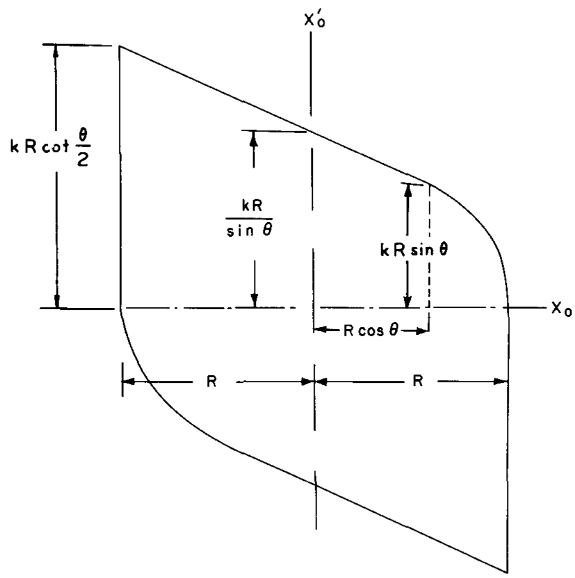
$$\sin\theta \sinh\theta < 1 + \cos\theta \cosh\theta < 2 + \sin\theta \sinh\theta$$

are not critical. They lead to the conditions

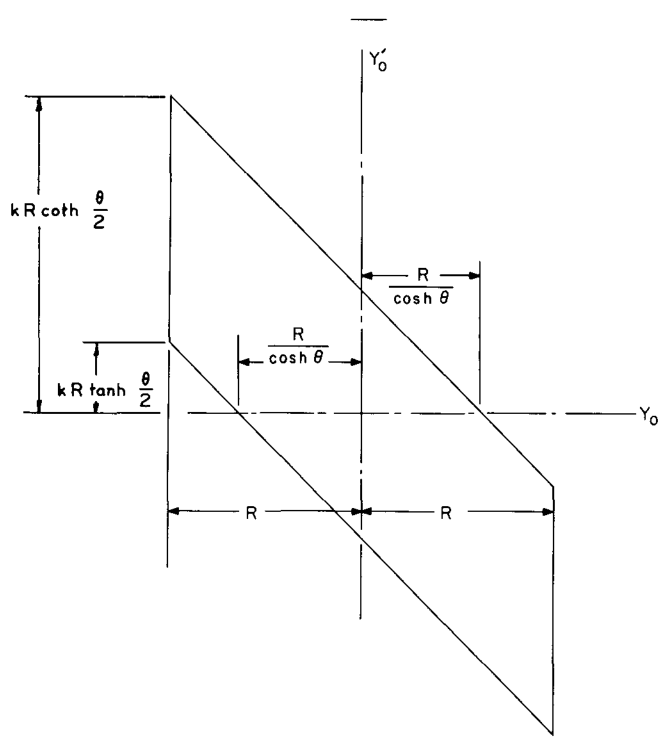
$$0.008 < \theta < 1.27$$

which will be satisfied in the vast majority of practical cases.

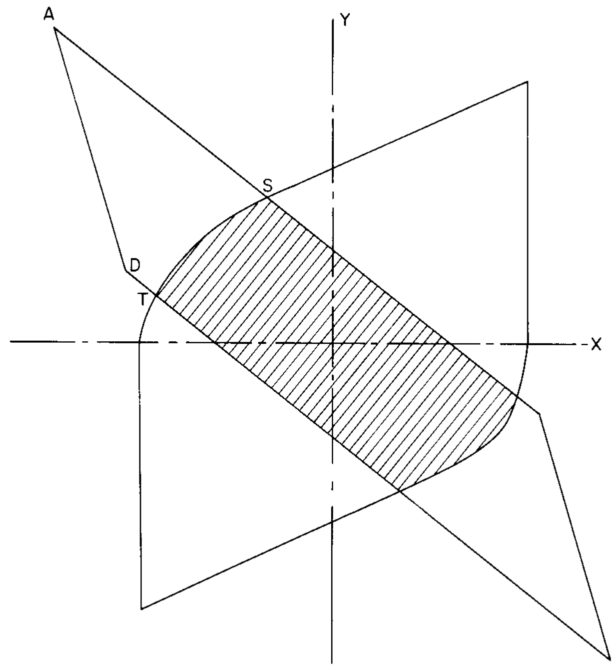
A more detailed account will be given in a CERN report.



(Figure 1) Phase space contour corresponding to the focusing plane at the entrance of the quadrupole.



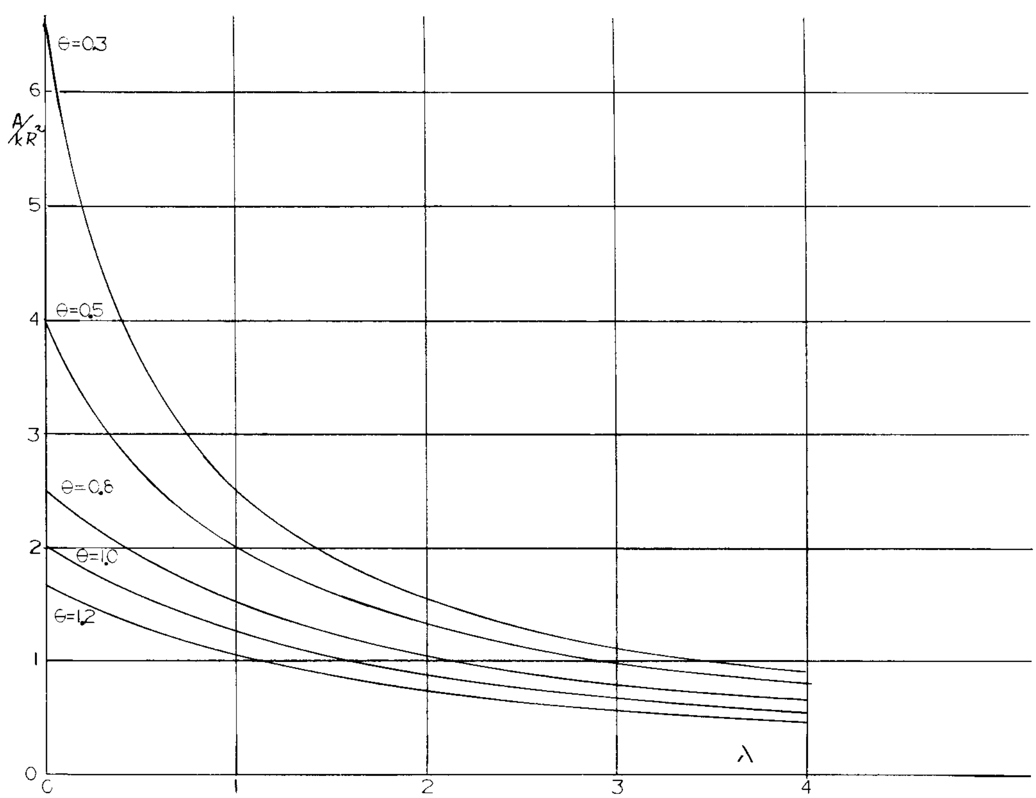
(Figure 2) Phase space contour corresponding to the defocusing plane at the entrance of the quadrupole.



$$A\left[X = -R\left(1 + kL\coth\frac{\theta}{2}\right),\ Y = kR\coth\frac{\theta}{2}\right]$$

$$D\left[X = -R\left(1 + kL\tanh\frac{\theta}{2}\right),\ Y = kR\tanh\frac{\theta}{2}\right]$$

(Figure 3) Phase space (shaded area) corresponding to the cd plane of a doublet, taken at the exit of the focusing element.



(Figure 4) Normalized acceptance A/kR^2 as a function of normalized drift length λ for various values of the parameter θ.