OSCILLATION MODES IN TWO DIMENSIONAL BEAMS

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ABSTRACT

The modes of oscillation of a uniform one-dimensional beam have been derived and studied by Sacherer and Smith. The present work is an extension of their work to two dimensions, using a matched Kapchinsky Vladimirsky beam as the unperturbed configuration. The method consists of obtaining two coupled integral equations in the perturbed phase space distribution and in the electric field. Solutions of these equations for different modes of oscillation are obtained, and the frequency spectrum of these modes is presented as a function of space charge intensity. It is shown that these modes correspond to surface distortions of the KV ellipsoidal surface in 4-dimensional phase space. The bearing of these results on space charge limits in linear and circular accelerators is discussed.

I. Introduction

The work of Kapchinsky and Vladimirsky[1] (KV) addressed itself to the behavior of a two-dimensional beam, under linear external restoring forces, whose distribution in (four-dimensional) phase space was on the surface of a hyperellipsoid. For this distribution the space charge forces are linear and the densities in the x-y, x-x', y-y', x'-y' projected spaces are uniform within an elliptical boundary. Their analysis led to a treatment of the envelope oscillations of such a beam and in particular to a study of the "breathing" and "quadrupole" modes of oscillation of the beam boundary.

It is clear that these two modes of oscillation are not the only ones possible. In particular, there is a coherent mode in which the beam oscillates as a rigid ellipse at a frequency corresponding to the external restoring force. These are but a few of the members of what should be a doubly infinite spectrum of modes. An understanding of these modes is important in predicting the response of the beam to a non-linear perturbing influence at an arbitrary frequency.

This problem has been studied in great detail by Sacherer[2] and Smith[3] for a one dimensional beam of uniform charge density. They obtain the eigen-modes of oscillation and their eigen-frequencies, and discuss the implications of external perturbing forces.

In the present paper we shall obtain corresponding results for the eigen-modes and eigen-frequencies of a two-dimensional KV beam. For simplicity we shall consider the non-relativistic case, although a relativistic treatment requires only minor modifications.

II. Formulation of the Linearized Equations

In the absence of collisions the equation satisfied by the distribution $f(x,y,u,v,t)$ is

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + \frac{F_x}{m}\frac{\partial f}{\partial u} + \frac{F_y}{m}\frac{\partial f}{\partial v} = 0 \tag{1}$$

where F_x and F_y are the components of the total force. We shall consider perturbations from the stationary KV distribution f_o, with the fields now being made up of the external field, the space charge field corresponding to the stationary distribution, and deviations therefrom. Specifically

$$f = f_o(x,y,u,v) + f_1(x,y,u,v,t) \tag{2}$$

$$\frac{F_x}{m} = -\nu_o^{\,2}x + \omega_p^{\,2}x + \frac{eE_x}{m}$$
$$\frac{F_y}{m} = -\nu_o^{\,2}y + \omega_p^{\,2}y + \frac{eE_y}{m} \tag{3}$$

where ν_o is the coherent oscillation frequency,

$$\omega_p^{\,2} = \frac{eI}{2\pi\varepsilon a^2 mv} = \frac{ne^2}{2m\varepsilon} \tag{4}$$

is the "plasma" frequency (n is the particle density in the beam), and $\vec{E}$ is the perturbed space charge field satisfying

$$\nabla\cdot\vec{E} = \frac{1}{\varepsilon}\iint du\,dv\,f_1. \tag{5}$$

The linearized equation for f_1 is

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - \nu^2 x\frac{\partial}{\partial u} - \nu^2 y\frac{\partial}{\partial v}\right)f_1 = -\frac{e}{m}\left(E_x\frac{\partial f_o}{\partial u} + E_y\frac{\partial f_o}{\partial v}\right) \tag{6}$$

where

$$\nu^2 = \nu_o^{\,2} - \omega_p^{\,2}. \tag{7}$$

Equations (5) and (6) are the homogeneous (in f_1, $\vec{E}$) coupled integro-differential equations which must be solved. To do this, we assume (and remove) the factor exp $(-i\omega t)$ in both f_1 and $\vec{E}$, and recognize that the operator on the left side of (6) corresponds to a known (sinusoidal) orbit, in which case one can write

$$\left(-i\omega + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - \nu^2 x\frac{\partial}{\partial u} - \nu^2 y\frac{\partial}{\partial v}\right)f_1 = Q(x,y,u,v) \tag{8}$$

$$f_1 = \frac{1}{\nu}\int_0^\infty ds\ e^{i\lambda s} Q(x',y',u',v'). \tag{9}$$

Here

$$\left.\begin{aligned} x' &= x\cos s - \frac{u}{\nu}\sin s \\ y' &= y\cos s - \frac{v}{\nu}\sin s \\ u' &= u\cos s + \nu x\sin s \\ v' &= v\cos s + \nu y\sin s \end{aligned}\right\} \tag{10}$$

$$\lambda = \omega/\nu \tag{11}$$

and the path of integration is properly deformed at $s=\infty$ to obtain convergence. A further simplification comes from the fact that f_o is a function of the single variable R^2 given by

$$R^2 = x^2 + y^2 + \frac{u^2}{\nu^2} + \frac{v^2}{\nu^2} \tag{12}$$

in which case

$$\frac{\partial f_o}{\partial u} = \frac{2u}{\nu^2} f_o'(R^2)\ ,\quad \frac{\partial f_o}{\partial v} = \frac{2v}{\nu^2} f_o'(R^2)\ . \tag{13}$$

This enables us to write

$$f_1 = -\frac{2e}{m\nu^3}\int_0^\infty ds\ e^{i\lambda s} f_o'(R^2)(u'E_x(x',y') + v'E_y(x',y')). \tag{14}$$

Since we are dealing with non-relativistic conditions, particularly for the first order oscillations, $\vec{E}$ is derivable from the scalar potential $G(x,y)$ with

$$\vec{E} = -\nabla G. \tag{15}$$

An integration of (14) by parts finally leads to

$$f_1 = \frac{2ie\lambda}{m\nu^2} f_o'(R^2)\int_0^\infty ds\ e^{i\lambda s} G(x',y') \tag{16}$$

$$\nabla^2 G(x,y) = -\frac{1}{\varepsilon}\iint du dv\ f_1(x,y,u,v) \tag{17}$$

where we have been able to extract $f_o'(R^2)$ from under the integral since R^2 is unaffected by the "rotation" in (10).

A note of caution is appropriate here. The term $f_o'(R^2)$ represents the highly singular derivative of a delta function, and our mathematical manipulations must be done with care. Furthermore, the assumption of $\exp(-i\omega t)$ dependence does not permit movement of the beam boundary. Rather we must consider charge building up on the (fixed) surface of the hyper-ellipsoid and will at a later point translate this to equivalent motion of the surface by proper consideration of the currents associated with f_1.

III. Solution of the Linearized Equation

It is difficult to derive the solutions directly from (16) and (17). Instead we have observed that polynomial solutions for G can be constructed and have generalized these to the statement that the solution can be written (in polar coordinates) as

$$G(r,\theta) = r^m \left\{ \begin{matrix} \cos m\theta \\ \sin m\theta \end{matrix} \right\} {}_2F_1(-j,\ m+j,\ m+1;\ r^2) \tag{18}$$

where $j = 0,1,2,\ \ldots,\ m = 0,1,2,\ldots$ (not including $j=m=0$), and where the radius r is in units of the unperturbed radius a. The frequency is given by the equation

$$\int_0^\infty ds\ e^{i\lambda s} \cos^m s\ {}_2F_1(-j,\ m+j,\ m+1;\ \cos^2 s) = -\frac{1}{i\lambda}\left[\delta_{jo} + \frac{(-1)^j m!\,j!\,\nu^2}{(m+j-1)!\,\omega p^2}\right] \tag{19}$$

Several steps have been taken to demonstrate the correctness of these assertions:

1) The usual series expansion for the hypergeometric function ${}_2F_1$ has been used in (16) to permit term by term integration. In the process one uses theorems like

$$\sum_k \frac{1}{(A+k)!(B-k)!(C+k)!(D-k)!} = \frac{(A+B+C+D)!}{(A+B)!(C+D)!(A+D)!(B+C)!} \tag{20}$$

2) The value of f_1 thus obtained is integrated to confirm the validity of (17) subject to (19).

3) The current is calculated from f_1, and the surface charge density is thus obtained. It is then confirmed that the discontinuity in ∇G appropriate to the surface charge is consistent with $G = r^{-m} e^{\pm im\theta}$ in the exterior region.

One further note is of interest at this point. The full distribution will now be of the form

$$f_o + f_1 \cong K[\delta(R^2 - a^2) + \bar{\varepsilon} g(x,y,u,v)\delta'(R^2 - a^2)] \tag{21}$$

where $\bar{\varepsilon}$ is an arbitrarily small constant, and the form of g is derived from (16) and (18). One can write an alternate form of (21)

$$f_o + f_1 \cong K\delta(R^2 - a^2 + \bar{\varepsilon} g(x,y,u,v)) \tag{22}$$

which shows that the eigen-modes take the form of distortions of the surface distribution in the four-dimensional phase space.

IV. Particular Solutions

Several special cases are of interest and are easily obtained from (18) and (19).

A. $j = 0,\ m = 1$

$$\left.\begin{array}{l} G = x \ , \ E_x = -1 \ , \ E_y = 0 \\ \text{or} \\ G = y \ , \ E_x = 0 \ , \ E_y = -1 \end{array}\right\} \tag{23}$$

$$\omega_{01}^{\ 2} = \lambda_{01}^{\ 2}\nu^2 = \nu^2 + \omega_p^{\ 2} = \nu_o^{\ 2} \ . \tag{24}$$

Clearly this case corresponds to a coherent oscillation at the proper frequency. The charge density $(\varepsilon\nabla\cdot\vec{E})$ is obviously unchanged.

B. $j = 0,\ m = 2$

$$G = x^2 - y^2 \ , \ E_x = -2x \ , \ E_y = 2y \tag{25}$$

$$\omega_{02}^{\ 2} = \lambda_{02}^{\ 2}\nu^2 = 4\nu^2 + \omega_p^{\ 2} = 4\nu_o^{\ 2} - 3\omega_p^{\ 2} \ . \tag{26}$$

This case corresponds to quadrupole beam oscillations; the frequency is the same as that derived from envelope considerations. The charge density is again unchanged.

C. $j = 1,\ m = 0$

$$G = 1 - r^2 \ , \ E_x = 2x \ , \ E_y = 2y \ , \ \nabla\cdot\vec{E} = 4 \tag{27}$$

$$\omega_{10}^{\ 2} = \lambda_{10}^{\ 2}\ \nu^2 = 4\nu^2 + 2\omega_p^{\ 2} = 4\nu_o^{\ 2} - 2\omega_p^{\ 2} \tag{28}$$

This case corresponds to the usual "breathing" oscillation; the frequency is the same as that derived from envelope considerations. As expected, in this case the perturbing charge density is constant corresponding to a uniform beam of oscillating radius.

D. $j = 2,\ m = 0$

$$\left.\begin{array}{l} G = 1 - 4r^2 + 3r^4 \ , \ E_x = -12x^3 - 36xy^2 + 8x \\ E_y = -12y^3 - 36x^2y + 8y \ , \ \nabla\cdot\vec{E} = 16(1 - 3r^2) \end{array}\right\} \tag{29}$$

$$\left.\begin{array}{l} (\lambda_{20}^{\ 2} - 4)(\lambda_{20}^{\ 2} - 16) = \dfrac{2\omega_p^{\ 2}}{\nu^2}(\lambda_{20}^{\ 2} + 2) \\ \omega_{20}^{\ 2} = \lambda_{20}^{\ 2}\nu^2 = 10\nu^2 + \omega_p^{\ 2} - \sqrt{36\nu^4 + 24\omega_p^{\ 2}\nu^2 + \omega_p^{\ 4}} \end{array}\right\} \tag{30}$$

where we have chosen the negative sign for the square root in order to have $\omega_{20} = 2\nu$ when $\omega_p = 0$. This mode of oscillation corresponds to a breathing mode with non-uniform density. In fact, there is a radial node at $r^2 = 2/3$, and, as the beam expands, the density increases for $r^2 < 2/3$ at the same time as it decreases for $r^2 > 2/3$.

For small $\omega_p^{\ 2}/\nu^2$ one finds

$$\omega_{20}^{\ 2} \cong 4\nu^2 - \omega_p^{\ 2} = 4\nu_o^{\ 2} - 5\omega_p^{\ 2} \ . \tag{31}$$

Comparison with (28) indicates that the shift in frequency for the (2,0) mode is 2.5 times that for the (1,0) mode.

E. Other j, m

In general one finds other modes with frequencies given for small ω_p^2/ν^2 by

$$\left.\begin{aligned} \omega_{jm}^2 &\cong \nu^2 + \alpha_{jm}\omega_p^2 = \nu_o^2 + (\alpha_{jm}-1)\omega_p^2\ , \quad m \text{ odd} \\ \omega_{jm}^2 &\cong 4(\nu^2 + \alpha_{jm}\omega_p^2) = 4(\nu_o^2 + (\alpha_{jm}-1)\omega_p^2),\quad m \text{ even} \end{aligned}\right\} \qquad (32)$$

Some of the values of α_{jm} are given in Table I.

Table I

α_{jm} vs j and m

m \ j	0	1	2
0	—	1/2	-1/4
1	1	-1/4	0
2	1/4	0	-5/64
3	1/4	-1/8	1/64
4	1/8	-1/32	-1/32
5	1/8	-5/64	1/64

V. Space Charge Limits in Synchrotrons

The simplistic calculation for the space charge limit in a synchrotron (small ω_p^2/ν^2) starts with (7) written in the form

$$\Delta\nu = \nu - \nu_o \simeq -\omega_p^2/2\nu_o \qquad (33)$$

and requires

$$|\Delta\nu| < \frac{\nu_{rot}}{4} \text{ or } \omega_p^2 < \frac{\nu_o\nu_{rot}}{4}\ , \qquad (34)$$

corresponding to the maximum distance to either a half-integral or integral stop band. Here ν_{rot} is the rotation frequency in the circular accelerator. If one takes into account other modes, one finds for odd m that $|\Delta\nu| < \nu_{rot}/4$ corresponds to

$$\omega_p^2 < \frac{\nu_o\nu_{rot}}{2(1-\alpha_{jm})} \qquad (35)$$

and for even m, that $|\Delta\nu| < \nu_{rot}/2$ corresponds to exactly the same equation. For the breathing mode (1,0) one has

$$\omega_p^2 < \nu_o\nu_{rot}\ , \qquad (36)$$

in agreement with the increased limit found by Lloyd Smith[4] based on the analysis of the envelope oscillations. However, the most restrictive limit is the one having the lowest (largest negative) value of α_{jm}; according to Table I there are two such cases: (2,0) and (1,1), for which

$$\omega_p^2 < \frac{2}{5}\nu_o\nu_{rot} \quad . \tag{37}$$

We therefore find that there are two modes of beam oscillation which imply that the space charge limit is lower (by 20%) than that predicted by the simple result in (34).

It might be wise to point out at this time that real beams do not in general have a precise KV distribution. In fact, the higher modes involve internal density variations, and are probably more sensitive to the details of the distribution than is the breathing mode. For this reason the limit in (37) may be somewhat unrealistic. Actual orbit calculations are required to see what the time limit is.

VI. Space Charge Limits in Linear Accelerators

The simplistic calculation for the space charge limit in a linear accelerator consists of the requirement that ν^2 be positive in (7), corresponding to stable oscillations for all individual particles. Such considerations do not take into account questions of stability which are the subject of a companion paper.[5]

One expects however, that an unstable condition for the KV beam would be reflected by an imaginary frequency for at least one beam mode. Examination of the mode frequencies indicates that almost all frequencies are real in the range $0 \le \omega_p^2 \le \nu_o^2$. The only exception found thus far is for the (2,2) mode discussed in Section IVD. From (30), one finds that

$$\begin{aligned} \omega_{20}^2 &\ge 0 \quad , \quad \omega_p^2 \le \frac{16}{17}\nu_o^2 , \\ \omega_{20}^2 &< 0 \quad , \quad \frac{16}{17}\omega_o^2 < \omega_p^2 < \nu_o^2 . \end{aligned} \tag{38}$$

Thus we expect that when the space charge intensity is sufficiently high, the KV distribution will become unstable. The growth rate predicted by (30) is extremely slow within the range of instability, reaching a maximum of $-\omega_{01}^2 \cong .023\nu_o^2$ when $\omega_p^2 \cong .97\,\nu_o^2$.

In Figures 1 and 2 we have plotted ω_{jm}^2/ν_o^2 vs ω_p^2/ν_o^2 for several of the interesting modes.

VII. Numerical Orbit Calculations

Orbit calculations have been performed for axially symmetric beams by R. Chasman and K. Crandall[6] to test the predictions of this paper. In particular, the starting distribution has been chosen to correspond with (22) for both the lowest breathing mode (1,0) and the next lowest (breathing) mode (2,0). In the case of the (1,0) mode, we find oscillations of the beam envelope which are in complete agreement with frequency predicted in (28) for values of ω_p^2 between 0 and ν_o^2. For the (2,0) mode we find oscillations of the beam envelope which are in excellent agreement with the frequency predicted by (30) for values of ω_p^2 between 0 and $.85\nu_o^2$. For values of ω_p^2 above $.85\nu_o^2$ the oscillations are erratic and less clearly defined. We have also searched for instability in the range $16/17 < \omega_p^2/\nu_o^2 < 1$. Although the evidence is not conclusive because of inherent numerical limitations associated with a finite but large number of particles undergoing a finite but large number of impulses, there is a suggestion of instability of the KV distribution for these values of ω_p^2/ν_o^2. These calculations are in the process of being improved; as yet we are unclear as to what is the most stable distribution in this range of ω_p^2/ν_o^2.

It may be well to point out that the intensity limit imposed by $\omega_p^2/\nu_o^2 < 16/17$ does not differ in a practical sense from the limit $\omega_p^2/\nu_o^2 < 1$. Our concern instead is that another distribution may be more stable. The KV distribution pushed to the limit may then lead to deterioration of the quality of the beam as the limit is approached.

VIII. Conclusions

We have derived the eigen-frequencies and eigen-modes for a two dimensional KV beam. The space charge limit in both circular and linear accelerators appears to be connected with the second azimuthally symmetric mode of beam oscillation, and is more restrictive than the usual space charge limit considerations. Agreement with numerical orbit calculations is good.

IX. Acknowledgement

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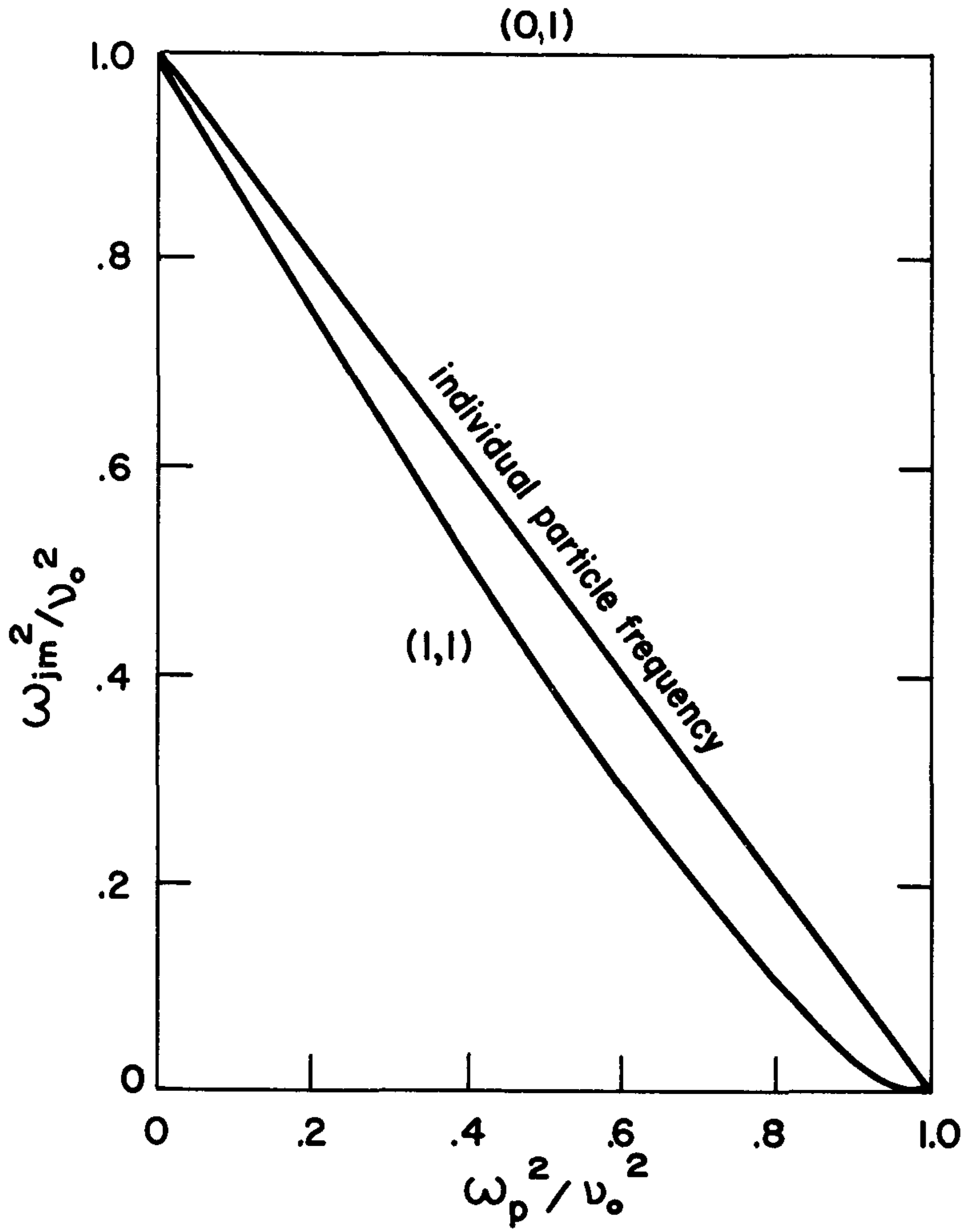

Fig. 1. Mode frequency vs beam intensity for various "odd" modes.

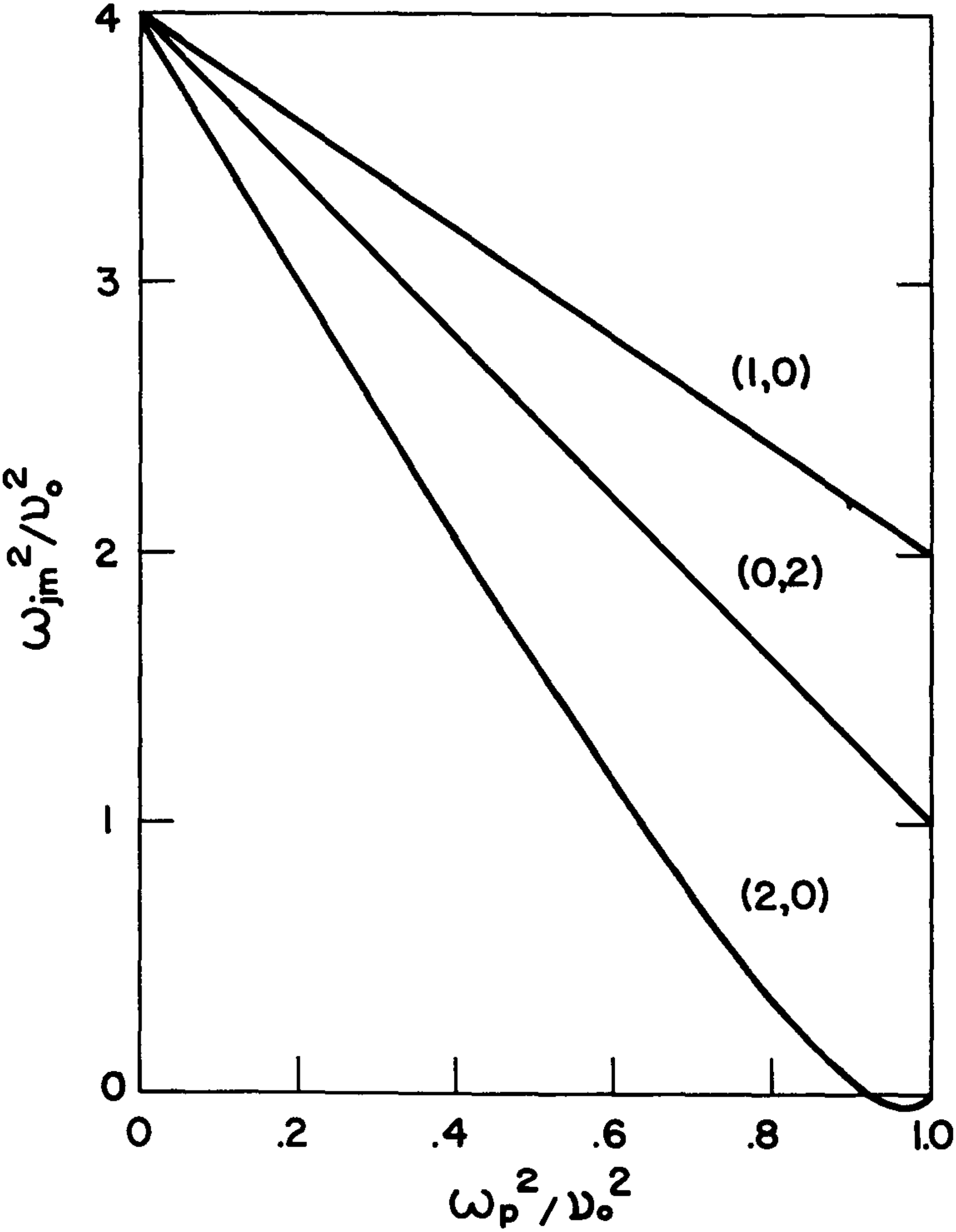

Fig. 2. Mode frequency vs beam intensity for various "even" modes.

DISCUSSION

(The discussion of this paper follows LCO-063, "Stability of Phase Space Distributions in Two Dimensional Beams" by R. L. Gluckstern, R. Chasman, and K. Crandall.)