A STATISTICAL APPROACH TO THE ESTIMATION OF BEAM SPILL*

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Summary

Computer codes, which transform a number of pseudo-particles through a simulated machine, form the backbone of accelerator design techniques. When minimization of beam loss within the machine is a primary design objective, it becomes necessary to consider beam behavior in detail, and questions of both the physics and computational aspects of the simulation are raised. Within the constraints of the former, the effectiveness of the latter can be increased for a given amount of computer resources by the use of statistical techniques. A statistical approach to determining the maximum beam size and hence required aperture of a machine is described and illustrated. The method draws upon statistical theory to treat the maximum radius attained by a finite group of particles passing through an accelerator as a statistical variable. Once the distribution of this variable is obtained, radius values can be found inside which a given percentage of the particles can be expected. Confidence bounds can be placed on these radii, and the results used to estimate the suitability of the accelerator apertures.

Introduction

The estimation of beam spill has taken on greater importance in recent years with the call for higher currents in accelerators. LASL interest in the problem arose from the need to estimate beam spill in the Fusion Materials Irradiation Test (FMIT) linac. Here three factors combine to make an accurate estimate necessary. These are the relatively high average current (100 mA cw), the acceleration of a deuteron beam with consequent higher radiation produced per lost particle, and the requirement for no remote handling during maintenance.

At Los Alamos, the PARMILA code is used to transform particles through the six-dimensional phase space of a simulated machine. Because practical computing considerations limit the number of particles that can be followed to a few hundred or a few thousand, compared to 10^8-10^9 in the real beam, the code cannot be expected to give good answers concerning the absolute outer boundary of the beam even if the mathematical formulation of particle movement were exactly correct.

The accuracy of the mathematical formulation is a problem in physics and code design. With any given model, however, the problems caused by following only a small sample can be alleviated by using statistical methods[1,2] to assess the position of the outer boundary.

The Statistical Approach

The parameter that determines beam spill is, in the final analysis, the maximum radius (r_{max}) assumed by any particle in the bunch. The maximum radius is dependent on the focusing strength, the rf defocusing in the gaps, space-charge (hence the beam current), the degree of mismatch, and the alignment and quality of the quadrupole magnets and drift tubes. All these factors are included in the PARMILA code model.

Figure 1 shows the probability density function (pdf) for the physical radius of a beam of particles randomly selected from a uniform distribution in the four-dimensional transverse phase space. An actual pdf histogram for a typical sample of 500 particles is superimposed. Such groups of particles are transported through the accelerator code, and the maximum radius is observed at suitable intervals. (In the strong-focusing system used as the example here, the beam is observed at the center of each quadrupole.) Although the

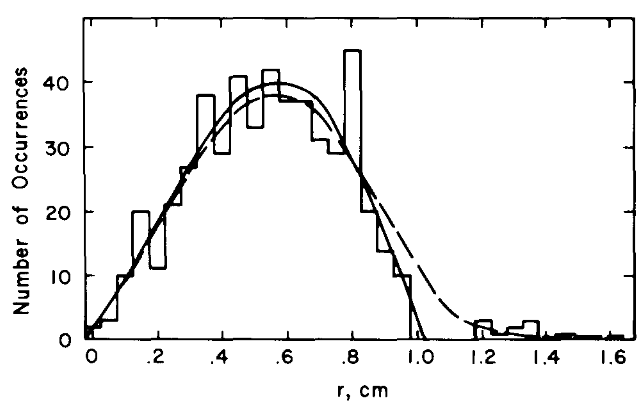

Fig. 1. The pdf for the radius of a typical input particle distribution is shown by the solid curve, along with the histogram resulting from an actual sample of 500 particles. The dashed curve is a hypothetical pdf observed in accordance with the output criterion. A particular r_{max} for the sample will occur.

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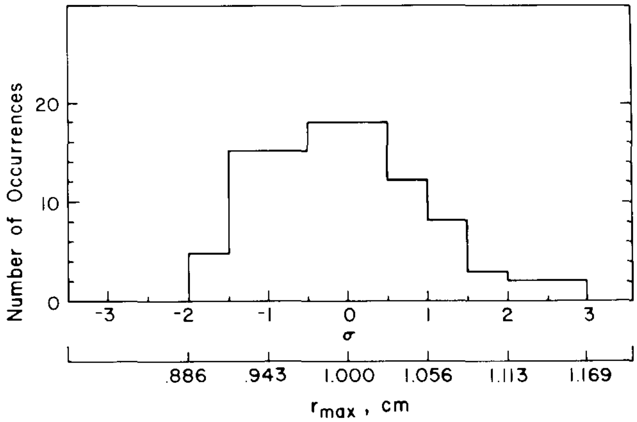

Fig. 2. Histogram of the maximum radius observed anywhere in a section of linear accelerator in 100 runs of 500 particles each. Scales unrelated to Fig. 1.

initial distribution is usually bounded, particles that are not properly accelerated or transported mav reach arbitrarily large radii, except that they would strike some limiting physical surface in a real machine. The r_{max}, observed at any particular point, or over a suitable length of machine that integrates over the various oscillation effects, also may be considered a random variable. A series of simulations, starting with different particle samples from the initial distribution, will yield a set of maximum radii with a probability distribution of their own, indicated in Fig. 2.

Let $f(r) = F'(r)$ be the pdf of the distribution of radii in the region of interest, then $F(r)$ is the probability that an observed radius is less than a certain r. The probability that n independent observations all fall short of r is then $F^n(r)$; i.e, this is the probability that r is the l.rgest among n independent observations. Let $\Phi_n(r_n)$ be the probability that the largest value falls short of r_n:

$$\Phi_n(r_n) = F^n(r_n) \quad .$$

Then the derivative,

$$\phi_n(r_n) = nF^{n-1}(r_n)f(r_n) \quad ,$$

is the distribution of the largest among n independent observations. These equations form the basis for an exact theory of extreme values, which proceeds to explore whether asymptotic distributions valid for large samples exist, their nature, how quickly they are approached, and how to estimate their parameters from sample data.

The parameters used to characterize the pdf of the extremes, $\phi(r)$, are the expected value of the extremes, u_n, and the parameters $\alpha_n = nf(u_n)$. It can be shown that:

$$\frac{du_n}{d(\log n)} = \frac{1}{\alpha_n} \quad ,$$

which indicates that $1/\alpha_n$ measures the increase of the expected largest value with the logarithm of the sample size. Distributions fall into three classes, depending on whether α_n increases, remains constant, or decreases with n. This characteristic indicates that the exponential function underlies the theory. Distributions of extremes may also be placed another way into three categories: those that are unlimited (in one or both directions) where all moments exist, unlimited distributions with only a finite number of moments, and limited distributions.

The distributions of the extremes share properties with their underlying population distribution: limited (or not) to left or right, possession of moments, and asymptotic behavior dependent on the behavior of the parent distribution. It is erroneous, however, to assume that the distribution of extremes is normal or tends to normal; most distributions of extreme values are skewed and remain that way in the asymptote. The few symmetric distributions to be found are not normal. It is seen that the empirical pdf in Fig. 2 is bounded to the left, skewed, and could be unlimited to the right.

As with other types of probability distributions, rules are derived for estimating the parameters, for estimates of extrapolated values, and for confidence bounds on the estimates. The appropriate probability paper can be derived; probability paper provides the simplest way to evaluate whether an observed distribution fits the theory (the data would plot on a straight line), and to make estimates. The fitted distribution provides a best guess as to how **an infinite** number of particles would behave. The effect of the finiteness of the original data is now apparent only in the confidence **bounds** on the fitted distribution. The fitted distribution can now be used to provide estimates of the maximum radius inside which a given percentage of maximum radii will be found.

Application

The linac data fit an extreme value distribution used by Weibull, which also finds wide use in reliability theory. The cumulative Weibull distribution is described by

$$F(r) = 1 - e^{\frac{-(r-\gamma)^{\beta}}{\alpha}} \quad ,$$

where α is called the scale parameter, β is the shape parameter, and γ is the location parameter.[3] Figure 3 shows the data of Fig. 2 plotted on Weibull probability paper. The variable $x = r - \gamma$ is plotted versus F. The points are seen to very closely approximate a

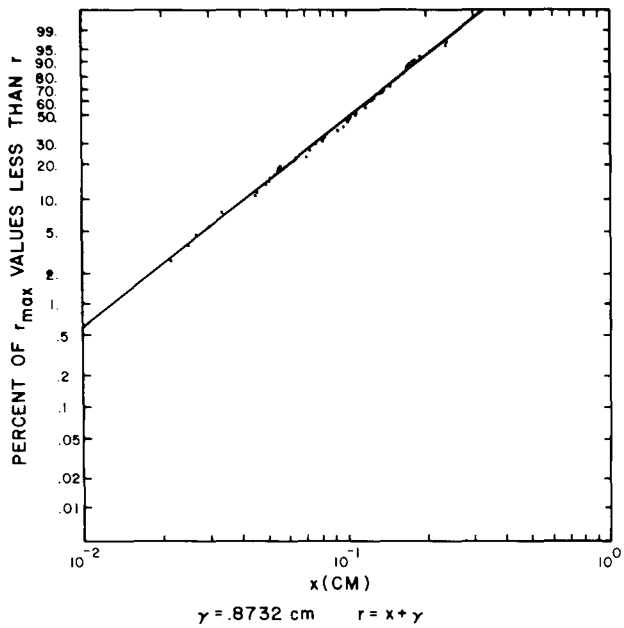

Fig. 3. One hundred values of r_{max} (from 100 PARMILA runs using independent input distributions of 500 particles each) plotted on Weibull coordinates. The straight line is the fitted Weibull distribution.

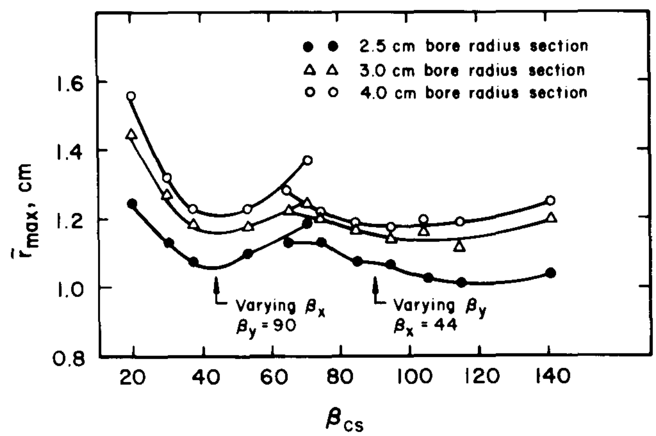

Fig. 4. Average r_{max} observed in 10 runs at each condition, in a study of the maximum beam size in lengths of the FMIT drift-tube linac having a certain bore size, as the match of the input beam is changed by varying the input beam size. β_x and β_y were varied independently.

straight line. Chi-square goodness-of-fit tests were made and show that the Weibull distribution provides a suitable description.

When good confidence bounds are required, the number of runs needed to determine the distribution will be fairly large. For parameter searches, however, as few as 10 runs can suffice. Estimates of the parameters of the distribution can be obtained from the plot; a computer program was written which accomplishes this by direct fitting of the data. Computer codes are being developed to produce maximum likelihood estimates of the parameters and to find the confidence bounds on the parameters and the distribution.

In the accelerator cases studies, the average value of the observed extremes is very close to the Weibull median value. Thus the mean $\bar{r}_{max}$ derived from a small number of runs, say 10, can be used as a quick measure to plot the effects of changes in the accelerating conditions. Figure 4 shows an example of the variation in $\bar{r}_{max}$ in sections of particular bore size of the FMIT linac as a function of how well the input beam is matched.

Another way of obtaining a quick estimate of the radius containing a given percentage of the maxima that would ever be observed, without fitting the distribution, is provided by median rank order statistics.[4] For example, with a sample of size 25, the largest of the 25 values corresponds to 97.3% of all maxima that would be observed. These points are shown on Fig. 5; each point corresponds to the largest of 25 runs. The effect of 0.25-mm random misalignment error is also shown. There is much more uncertainty in Fig. 5 than in Fig. 4, because the 97.3% estimate is based on less data than is the mean.

Conclusion

The observed extreme values from linear accelerator computer programs fit the statistical theory for extreme values very well. This theory should be used to develop estimates for beam spill in these machines. Although better models of the particle physics may also be needed, many of the effects of finite sample sizes are avoided by using the statistical approach. Work is progressing on identifying more precisely the characteristics of the particular distributions involved in the linac problem, programs for fitting data and calculating confidence bounds, and application of the results to judgments concerning the appropriate safety factors in accelerator design.

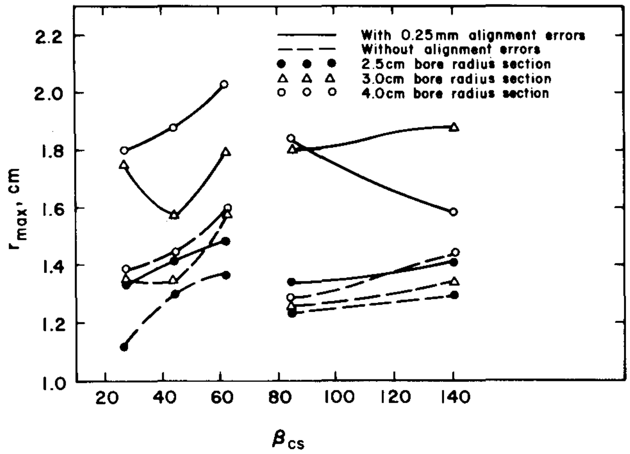

Fig. 5. For the matching experiment shown in Fig. 4, the maximum r_{max} in each series of 25 runs for each condition is plotted. These values, within a known uncertainty, should be greater than or equal to 97.3% of all maximum radii that would be observed as the number of runs increased.

References

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