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FOCUSING OF HIGH CURRENT BEAMS IN CONTINUOUSLY ROTATED QUADRUPOLE SYSTEMS

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The recent development of high field permanent magnetic quadrupoles by Halbach, opens up their potential use in drift tube linacs, beam transport lines, etc. In order to retain some ability to adjust the focusing strength of a permanent quadrupole beam line, modification is proposed of the standard + − + − system to a system where adjacent quadrupoles are rotated (axially) with respect to one another by a constant angle, which can be adjusted to yield different focal strengths. The resulting motion in the two transverse directions is coupled, and requires a 4×4 matrix analysis to solve. However, a "smoothed" system consisting of continuously rotated quadrupoles can be solved analytically without invoking the usual alternating gradient matrix analysis. This has been done in this paper, which also extends the analysis to include properly matched, self-consistent, K-V space-charge distributions.

Introduction

The recent development of high field permanent magnetic quadrupoles[1] makes possible their use in transverse focusing applications such as transport lines, drift tube linacs, etc., without costly systems to supply magnet power and cooling. The simple means of adjusting focusing by changing the magnet excitation, however, is no longer available and must be replaced by some other mechanism which will permit changes for beam matching, space-charge defocusing, etc. The possibility explored in this paper, is the rotation of the quadrupoles about the beam axis. Although this couples the two transverse directions, the degree of rotation between successive magnets in a periodic system provides an adjustment of the focal strength in the transverse directions. The general features of a rotated quadrupole focusing system will first be discussed, and then a continuously rotated system analyzed in detail, including the presence of space-charge in a two-dimensional uniform beam of elliptical cross-section.

General Analysis

In a system of quadrupoles periodically spaced, and rotated with respect to one another by a fixed angle, the 4×4 matrices corresponding to the action of the quadrupole, to a drift, and to an rf defocusing impulse, are given by:

$$M_{quad} = \begin{pmatrix} \cos\theta & \frac{\sin\theta}{k} & 0 & 0 \\ -k\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cosh\theta & \frac{\sinh\theta}{k} \\ 0 & 0 & k\sinh\theta & \cosh\theta \end{pmatrix} \qquad (1)$$

$$M_{drift} = \begin{pmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2)$$

$$M_{rf} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \hat{\Delta} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \hat{\Delta} & 1 \end{pmatrix} \qquad (3)$$

where the transverse vector has components x, x', y, y' along the principal axes of the quadrupole. Here the equations of motion are given by:

$$\frac{d^2x}{ds^2} = -k^2x, \quad \frac{d^2y}{ds^2} = k^2y, \qquad (4)$$

and $\theta = k\ell$, where ℓ is the magnet length. The drift length is L and the rf impulse is given by:

$$\frac{x'}{x} = \frac{y'}{y} = \hat{\Delta}. \qquad (5)$$

The matrix appropriate to the traversal from a given axial location to the corresponding location one magnet period later, consists of the product of matrices of the form in (1), (2), (3) followed (or preceded) by the matrix

$$M_{rot} = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & \cos\alpha & 0 & \sin\alpha \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \qquad (6)$$

representing rotation by an angle α. Obviously, the vector now corresponds to the rotated coordinate system.

The transverse dynamics are now controlled by repeated multiplication by the one-cell matrix M, a process which is most easily accomplished by diagonalization of M. For example, if

$$SMS^{-1} = D \qquad (7)$$

is a diagonal matrix, then

$$M^n = S^{-1} D^n S. \qquad (8)$$

The motion will then be stable if the diagonalized matrix is of the form:

$$D = \begin{pmatrix} e^{i\sigma_1} & 0 & 0 & 0 \\ 0 & e^{-i\sigma_1} & 0 & 0 \\ 0 & 0 & e^{i\sigma_2} & 0 \\ 0 & 0 & 0 & e^{-i\sigma_2} \end{pmatrix} \qquad (9)$$

with σ_1 and σ_2 being real. If $\xi_i = (u,u',v,v')$ are components of the rotated vector, then one eventually can write

$$(\xi_i)^{(n)} = A_{1i}\cos n\sigma_1 + A_{2i}\sin n\sigma_1 + A_{3i}\cos n\sigma_2 + A_{4i}\sin n\sigma_2 \qquad (10)$$

where A_{ji} are (real) parameters determined by the focusing configuration. Clearly (10) can be solved for $\cos n\sigma_1$, $\sin n\sigma_1$, $\cos n\sigma_2$, $\sin n\sigma_2$ as linear combinations of the ξ_1. Forming $\cos^2 n\sigma_1 + \sin^2 n\sigma_1 = 1$ and $\cos^2 n\sigma_2 + \sin^2 n\sigma_2 = 1$ leads to two separate quadratic invariants involving the transverse displacements and angles. These invariants can then be used to evaluate maximum displacements and angles in terms of the initial displacements and angles. Needless to say, the algebra is straightforward but tedious. Some simplification may occur because the matrix M is symplectic and has a unit determinant. In any event, computer calculations are feasible.

Continuously Rotated Quadrupoles

As a special model, a continuously rotated quadrupole system will be considered for which the analytic calculation becomes tractable.[2] Although the details may differ from the cell-by-cell calculation outlined in the previous section, patterns of stability, variation of amplitude functions, matching requirements, and the effect of space would be expected to exhibit the same general features. For this reason, the calculations for the continuously rotated system will serve as a guide to the organization and interpretation of the computations for the cell-by-cell case which must ultimately be carried out.

Consider a quadrupole magnet system whose transverse axes rotate at an angular rate β per unit length along the axis. Since the free-space focusing force must be derivable from a potential satisfying Laplace's equation, the general form of this potential will be

$$\phi = \text{const}\ I_2(\beta r)\cos 2(\theta-\beta s). \tag{11}$$

Near the axis, only the lowest power of r^2 will contribute, and the equations for the transverse motion will be

$$\begin{aligned} x'' &= -Kx\cos 2\beta s - Ky\sin 2\beta s - x\Delta \\ y'' &= -Kx\sin 2\beta s + Ky\cos 2\beta s - y\Delta \end{aligned} \tag{12}$$

where K is related to the quadrupole strength, and Δ represents the (smoothed) rf defocusing gradient.

Let the transverse displacements to the rotating system be transformed in such a way that

$$\begin{aligned} u &= x\cos\beta s + y\sin\beta s \\ v &= -x\sin\beta s + y\cos\beta s . \end{aligned} \tag{13}$$

Differentiating (13), yields:

$$\begin{aligned} u' &= x'\cos\beta s + y'\sin\beta s + \beta v \\ v' &= -x'\sin\beta s + y'\cos\beta s - \beta u . \end{aligned} \tag{14}$$

Differentiating (14), gives:

$$\begin{aligned} u'' &= 2\beta v' + \beta^2 u + x''\cos\beta s + y''\sin\beta s \\ v'' &= -2\beta u' + \beta^2 u - x''\sin\beta s + y''\cos\beta s . \end{aligned} \tag{15}$$

Using (12) in Eq. (15) produces:

$$\begin{aligned} u'' &= 2\beta v' + \beta^2 u - Ku - u\Delta \\ v'' &= -2\beta u' + \beta^2 v + Kv - v\Delta \end{aligned} \tag{16}$$

in which the βs dependence has remarkably disappeared from the coefficients. What started out in (12) as two coupled Mathieu equations, have simplified to two coupled linear differential equations with constant coefficients in (16).

The solutions of (16) are obviously of the form $\exp(\pm ips)$, $\exp(\pm iqs)$, where $z = p^2, q^2$ are the two solutions of

$$(z+\beta^2-K-\Delta)(z+\beta^2+K-\Delta) = 4\beta^2 z, \tag{17}$$

namely

$$\begin{aligned} z_1 &\equiv p^2 = \beta^2 + \Delta + \omega \\ z_2 &\equiv q^2 = \beta^2 + \Delta - \omega, \end{aligned} \tag{18}$$

where

$$\omega = (K^2 + 4\beta^2\Delta)^{1/2}. \tag{19}$$

Clearly, stability requires that p^2 and q^2 each be real and positive. This translates into a condition on K given by

$$2\beta\sqrt{-\Delta} < K < \beta^2 - \Delta \tag{20}$$

which is represented by the shaded region in Fig. 1, a familiar shape in strong focusing systems.

In order to explore amplitude considerations, the general solutions of (16), will now be constructed. These can be written in the form

$$\begin{aligned} u &= A\cos P + \lambda B\sin Q \\ v &= -\sigma A\sin P + B\cos Q \end{aligned} \tag{21}$$

where $P = ps + a$, $Q = qs + b$ and

$$\lambda = \frac{2\beta^2+K-\omega}{2\beta q} = \frac{2\beta q}{2\beta^2-K-\omega} \tag{22}$$

$$\sigma = \frac{2\beta^2-K+\omega}{2\beta p} = \frac{2\beta p}{2\beta^2+K+\omega}. \tag{23}$$

From (21),

$$u' = -pA\sin P + \lambda qB\cos Q \tag{24}$$

$$v' = -\sigma pA\cos - qB\sin Q. \tag{25}$$

Elimination of the terms in P,Q leads to the two quadratic invariants

$$\frac{(u'-\lambda qv)^2}{\mu^2} + \frac{(\lambda v'+qu)^2}{\nu^2} = A^2 \tag{26}$$

$$\frac{(\sigma u'-pv)^2}{\mu^2} + \frac{(v'+\sigma pu)^2}{\nu^2} = B^2 \tag{27}$$

where $\mu = p - \lambda q\sigma$, $\nu = \lambda\sigma p - q$. The maximum radial excursion

$$R^2 = (x^2+y^2)_{max} = (u^2+v^2)_{max} \tag{28}$$

is then determined from (26) and (27) in terms of the focusing parameters and the initial conditions. It can be shown, after some algebra, that R is the larger of

$$\lambda A + B \text{ or } A + \sigma B.$$

Self-Consistent Phase Space Distribution/Space-Charge

It is well known that any function of the constants of the motion yields a stationary distribution in phase space. In this case, the constants of the motion are given in (26) and (27).

In order to apply this principle to the present case with space-charge, phase space distribution must be restricted to one like the Kapchinsky-Vladimirsky (K-V) distribution for which the charge distribution is uniform and the space-charge force is linear. In the present case this suggests a phase space distribution of the form:

$$f(u,u',v,v') = \text{const}\ \delta[M\,A^2(26) + N\,B^2(27) - 1] \quad (29)$$

where $A^2(26)$ and $B^2(27)$ are written in terms of u, u', v, v' as in (26) and (27). The space-charge distribution can be obtained directly by integrating over u' and v', by appropriately completing squares in $MA^2 + NB^2$. The result is the uniform elliptical distribution

$$\rho(u,v) = \rho_o\{{1 \atop 0}\}, \text{ for } \frac{MNu^2}{M\lambda^2+N} + \frac{MNv^2}{M+N\sigma^2}\{{< \atop >}\}1, \quad (30)$$

where the axes of the ellipse remain clearly oriented along the principal axes of the quadrupole.

It must be verified that the forces appropriate to an elliptical space-charge distribution have been used in calculating the equations of motion. The semi-major and semi-minor axes are given from (30) by

$$a^2 = \frac{(M\lambda^2+N)}{MN}, \quad b^2 = \frac{(M+N\sigma^2)}{MN}. \quad (31)$$

Since the fields within a uniform elliptical distribution with charge per unit length τ are given by

$$E_x = \frac{4\tau}{4\pi\varepsilon_o}\ \frac{x}{a(a+b)}, \quad E_y = \frac{4\tau}{4\pi\varepsilon_o}\ \frac{y}{b(a+b)} \quad (32)$$

Equation (16) can be rewritten as

$$\begin{aligned} u'' &= 2\beta v' + \beta^2 u - K_m u - u\Delta_{rf} + u\Delta_a \\ v'' &= -2\beta u' + \beta^2 u + K_m u - u\Delta_{rf} + u\Delta_b \end{aligned} \quad (33)$$

where K_m is now identified only with the actual magnetic gradient, Δ_{rf} is the rf component of Δ, and Δ_a, Δ_b are the space-charge terms, derived from (32), given by

$$a\Delta_a = b\Delta_b = \frac{120\text{ ohms } eIc}{Mv^3(a+b)}. \quad (34)$$

Equation (33) returns to the original form (16) if one makes the identification

$$K = K_m + \left(\frac{\Delta_b - \Delta_a}{2}\right) \quad (35)$$

$$\Delta = \Delta_{r-f} - \left(\frac{\Delta_b + \Delta_a}{2}\right). \quad (36)$$

These expressions are easy to understand. Equation (36) represents the enhancement of the rf defocusing by the (average) space-charge defocusing. Equation (35) represents the contribution to the magnetic quadrupole gradient from the eccentricity of the elliptical space-charge.

Phase Space Projections, Transverse Emittance

The appropriate momenta to use for the x and y phase space projections, are the canonical momenta derived from the Hamiltonian representation of (16), namely

$$\xi = u' - \beta v, \quad \eta = v' + \beta u. \quad (37)$$

In terms of ξ and η, the invariants (26) and (27) can be written as

$$\begin{aligned} \frac{(\xi+\beta\varepsilon v)^2}{\mu^2} + \frac{\lambda^2(\eta-\beta\delta u)^2}{\nu^2} &= A^2 \\ \frac{\sigma^2(\xi-\beta\delta v)^2}{\mu^2} + \frac{(\eta+\beta\varepsilon u)^2}{\nu^2} &= B^2 \end{aligned} \quad (38)$$

where

$$\varepsilon = \frac{\omega - K}{2\beta^2}, \quad \delta = \frac{\omega + K}{2\beta^2}. \quad (39)$$

Integration of (29) over v and η yields the u, ξ phase space projection (the x, x' projection at s = 0) as an ellipse with boundary

$$\frac{MNx^2}{M\lambda^2+N} + \frac{MNx'^2}{(\delta^2\sigma^2N+\varepsilon^2M)\beta^2} = 1. \quad (40)$$

Similarly, the v, η projection (the y, y' projection at s = 0) is

$$\frac{MNy^2}{M+N\sigma^2} + \frac{MNy'^2}{(\delta^2\lambda^2M+\varepsilon^2N)\beta^2} = 1. \quad (41)$$

The emittances corresponding to (40) and (41) are

$$\frac{W_x}{\pi} = \frac{(M\lambda^2+N)^{1/2}(\varepsilon^2M+\delta^2\sigma^2N)^{1/2}\beta}{MN} \quad (42)$$

$$\frac{W_y}{\pi} = \frac{(M+N\sigma^2)^{1/2}(\delta^2\lambda^2M+\varepsilon^2N)^{1/2}\beta}{MN}. \quad (43)$$

In the special case $W_x = W_y$, one finds

$$M\lambda = N\sigma \quad (44)$$

and

$$\frac{a}{b} = \left(\frac{\lambda}{\sigma}\right)^{1/2} = \left(\frac{\beta^2+K-\Delta}{\beta^2-K-\Delta}\right)^{1/4}. \quad (45)$$

In this case, lines of constant a/b are straight lines in Fig. 1, given by

$$\frac{K}{\beta^2} = \left(1 - \frac{\Delta}{\beta^2}\right)\left(\frac{a^4-b^4}{a^4+b^4}\right). \quad (46)$$

Selection of Parameters

A sequence by which the parameters can be selected is as follows:

1) The parameters Δ/β^2 and K/β^2 are selected so that the "operating point" will lie comfortably within the region of stability in Fig. 1. The known extremes of Δ_{rf} then suggest a desirable value for β, from which the value of K follows.

2) The parameters λ, σ, ε, δ are functions only of Δ/β^2 and K/β^2, and are therefore determined. From the predetermined emittances W_x and W_y, one finds the necessary values of M and N, using (42) and (43).

3) The matched beam parameters a, b and the corresponding angles can now be found from (31), (40) and (41).

4) Equation (36) can now be used to determine the movement of the operating point in the presence of space-charge, and (35) will give the necessary magnetic quadrupole strength.

5) Equation (29) describes the matched four-dimensional phase space, which, in effect, requires some restrictive correlations between the two-dimensional projections in (42) and (43).

Summary

It is possible to design a periodic transport system, consisting of quadrupoles rotated by a fixed angle relative to their nearest neighbors, which will provide for focusing in the two transverse directions. The general analysis can be carried out by means of 4×4 matrices, and the focal strength of the configuration can be adjusted by increasing or decreasing the angle of rotation between adjacent quadrupoles.

The analysis has been carried out in detail for a simplified model consisting of a continuously rotated quadrupole system, including space-charge, in the form of a matched K-V beam. Simple formulas have been derived for the stability region, the amplitude variations, and the transverse matching requirement necessary to maintain a self-consistent equilibrium distribution. These results should serve as a useful guide in the design of a periodic transport system consisting of rotated quadrupoles.

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References

1. K. Halbach, LBL-8906 (1979); also private communication.
2. This calculation has been carried out earlier for low space charge. See, for example, L. Teng, Argonne Rpt. ANLAD-55 (Feb. 1959); G. Salardi, et al., Nucl. Instr. & Meth. 59, 152 (1968); S. Ohnuma, TRIUMF Rpt. TRI-69-10 (1969); R. M. Pearce, Nucl. Inst. & Meth. 83, 101 (1970).

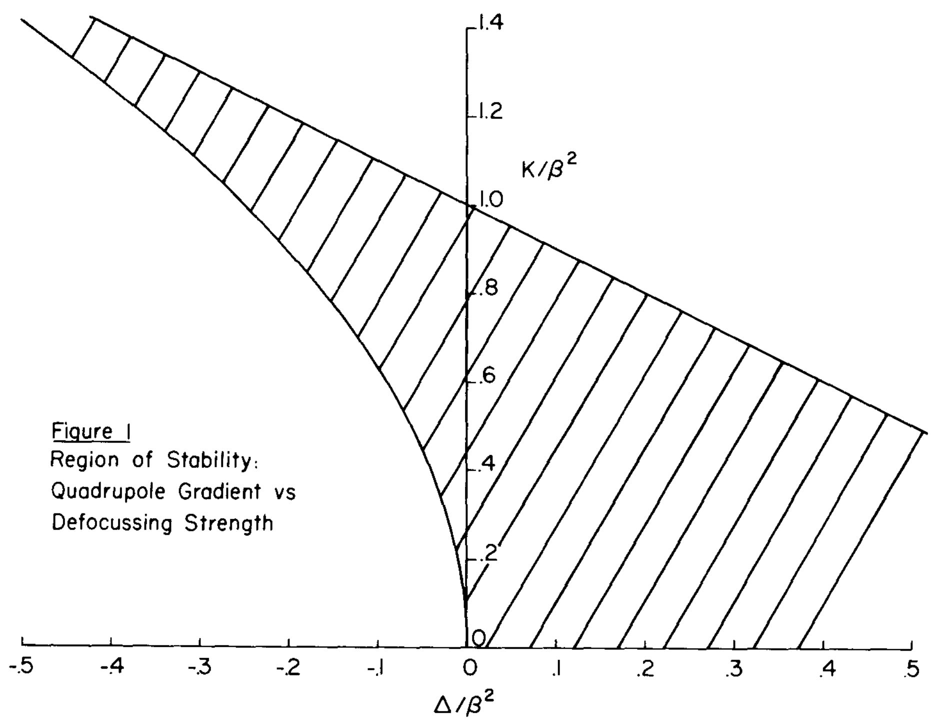

Figure 1
Region of Stability:
Quadrupole Gradient vs
Defocussing Strength