OCTUPOLE FOCUSING IN TRANSPORT AND ACCELERATION SYSTEMS*

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Summary

The radio-frequency quadrupole (RFQ) linac[1] is capable of accelerating high-current, low-velocity ion beams. In accelerator systems comprising an RFQ and higher velocity accelerating structures, the current bottleneck still typically occurs within the RFQ. This limiting current is quite high in most cases, but linacs with even higher currents may be required in the future. We have begun a study of higher multipole systems to determine their capability for focusing and accelerating very high currents. We have chosen first to examine a radio-frequency octupole (RFO) transport system, and have developed a smooth-approximation analytical description that includes the conditions for input radial matching of a zero space-charge beam. Further, we have constructed a multiparticle beam-dynamics simulation program that accepts the low-current matched beam and gradually increases the beam current as it is transported. This results in a matched high-current beam, and the procedure can be used to determine the saturation-current limit of a periodic octupole system. As expected, at high currents the beam develops a hollow radial distribution that reduces the space-charge defocusing; initial results show that high currents can be transported. For acceleration, we have formulated the design parameters for a section of RFO linac, including the potential function, acceleration, and focusing efficiencies, and the geometry of the radially modulated pole tips.

Introduction

We begin our study with an examination of the equations of motion for an RFO transport system. In the static limit, otherwise described as a quasi-static approximation, an octupole scalar potential can be written in cylindrical coordinates (r, Ψ, s) as

$$U = \frac{V}{2}\left(\frac{r}{a}\right)^4 \cos 4\Psi \sin \omega t \quad , \tag{1}$$

where V is the peak potential difference between adjacent poles, ω is the angular frequency of the time-varying voltage on each pole, and a is the radial aperture (see Fig. 1). Instead of time we use $z = s/L$ as the independent variable, where $L = 2\pi v/\omega$ and $v = ds/dt$. We obtain for the nonrelativistic equations of motion

$$x'' + B\,(x^3 - 3xy^2) \sin 2\pi z = 0 \quad , \tag{2}$$

and

$$y'' + B\,(y^3 - 3yx^2) \sin 2\pi z = 0 \quad , \tag{3}$$

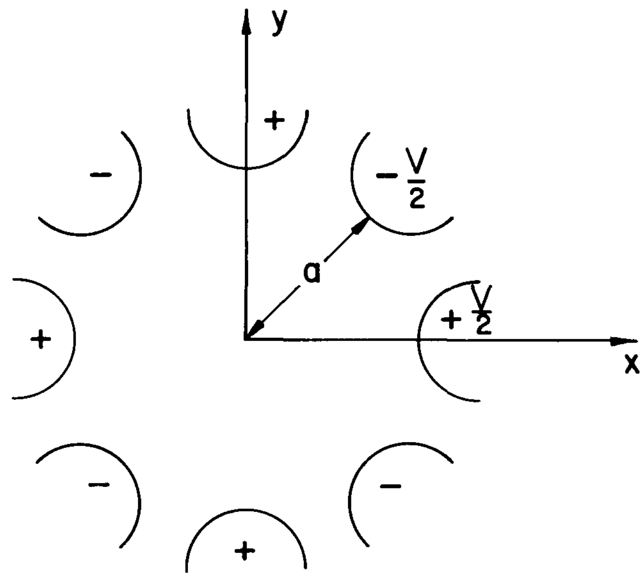

Fig. 1. Radio-frequency octupole cross section (unmodulated vanes).

where for a particle of charge q, and mass m, we define

$$B = \frac{2qV}{mv^2}\left(\frac{L}{a}\right)^4 . \tag{4}$$

The displacements x and y are dimensionless quantities, defined as the ratio of the actual displacement to the focusing period L. We use the convenient notation that $x'' = d^2x/dz^2$ and $y'' = d^2y/dz^2$.

The RFO Smooth-Approximation Solution

By analogy, with the smooth-approximation method for quadrupole focusing,[2] we assume solutions of the form

$$x = X(1 + u) \quad , \tag{5}$$

$$y = Y(1 + w) \quad , \tag{6}$$

where the functions X and Y, and their first and second derivative, are assumed to vary slowly enough to be considered constant over a period L. As with x and y, we define X and Y to be the dimensionless ratios of actual displacement to the period L. The functions u and w are assumed to be periodic with period L, and both $u \ll 1$ and $w \ll 1$. Furthermore, the mean values over a cell of u and w, and their first and second derivatives, are assumed to vanish. Thus, the solution is assumed to consist of a product of a slowly varying function times a rapidly varying periodic function.

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The slowly varying functions X and Y constitute what may be referred to as the smooth solution. After substituting the assumed solutions Eqs. (5) and (6) into the equations of motion Eqs. (2) and (3), and using the above approximations, we have obtained

$$u = \frac{B}{(2\pi)^2}(X^2 - 3Y^2)\sin 2\pi z \quad , \tag{7}$$

and

$$w = \frac{B}{(2\pi)^2}(Y^2 - 3X^2)\sin 2\pi z \quad . \tag{8}$$

Then, substitution of Eqs. (7) and (8) into the equations of motion averaged over a period, yields smoothed differential equations in X and Y:

$$X'' + GX(X^2 + Y^2)^2 = 0 \quad , \tag{9}$$

and

$$Y'' + GY(X^2 + Y^2)^2 = 0 \quad , \tag{10}$$

where G is given by

$$G = \frac{3B^2}{8\pi^2} \quad . \tag{11}$$

The equations of motion are coupled in general. For motion in the x,z plane, the smoothed uncoupled equation of motion is

$$X'' + GX^5 = 0 \quad . \tag{12}$$

This equation can be integrated twice to give an equation for smoothed phase advance per focusing period, σ_0, of uncoupled motion, which is

$$\sigma_0 = \frac{BX_m^2}{\sqrt{32}\;\Gamma} \; , \tag{13}$$

where X_m is the smoothed amplitude. The symbol Γ represents a ratio of gamma functions; the numerical value is given approximately as

$$\Gamma = \frac{\Gamma(7/6)\;\Gamma(1/2)}{\Gamma(2/3)} \simeq 1.2143 \quad . \tag{14}$$

From Eq. (13) we see that the phase advance depends not only on the focusing strength B, but also on the particle amplitude X_m.

We have compared the predictions of Eq. (13) with the phase advance obtained by using zero crossings of the numerical integration of Eq. (2) for uncoupled motion (y = 0), and find that the smooth approximation predicts the uncoupled phase advance to within a few per cent for phase-advance values up to 30°. The numerical-integration results also show that the uncoupled motion becomes unstable for phase-advance values equal or near to 30°, a result that has been independently discovered by Laslett.*

*L. J. Laslett, Lawrence Berkeley National Laboratory, personal communication, July 1981.

The RFQ Current Limit

We have searched for a simple model for a smoothed charge distribution within the beam, which we can use to represent the beam in an extreme space-charge regime. In the following discussion we present the motivation behind this model, as well as the resulting formulas.

The smoothed equation of motion can be expressed in cylindrical coordinates. If a smoothed radial coordinate R is defined by $R^2 = X^2 + Y^2$, then the smoothed radial equation of motion for a particle, subjected only to the applied octupole force, can be written as

$$R'' - \frac{J^2}{R^3} + GR^5 = 0 \quad , \tag{15}$$

where J is a constant of motion, proportional to the angular momentum. The second term constitutes a centripetal force, and the third term represents the applied octupole force. To represent the effect of the internal space-charge defocusing, we assume that an additional smoothed space-charge term, which depends upon R, can be added to Eq. (15). In an extreme space-charge limit, where the space-charge defocusing is barely balanced by the applied octupole focusing force, we assume that the first two terms can be ignored. Then we might expect that the beam charge would distribute itself, so as to minimize the free energy, by generating a space-charge term, which is balanced by the third term in Eq. (15) (the applied octupole force).

We are then led to assume a cylindrically symmetric charge model, where the smoothed space charge term has the same R^5 dependence as does the applied force. Gauss's law can be applied to the charge distribution, to yield an expression for the radial space-charge electric field given as

$$E = \frac{IR^5}{2\pi e_0 v R_m^6 L} \quad , \tag{16}$$

where I is the beam current and R_m is the maximum smoothed amplitude (the smoothed beam radius). The charge density is given as

$$\rho = \frac{3IR^4}{\pi v R_m^6 L^2} \quad . \tag{17}$$

This charge density is zero at R = 0 and increases strongly with radius R.

Within the context of this model, the smoothed equations of motion, (9) and (10), are modified to give

$$X'' + G(1 - \mu)\, X\, (X^2 + Y^2)^2 = 0 \quad , \tag{18}$$

and

$$Y'' + G(1 - \mu)\, Y\, (X^2 + Y^2)^2 = 0 \quad , \tag{19}$$

where

$$\mu G = \frac{qIZ_0 f^3}{2\pi mc^2\beta^3}\left(\frac{L}{a}\right)^6 \quad , \tag{20}$$

and μ is interpreted as a ratio of space charge to focusing force. We have introduced $\beta = v/c$, and $Z_0 = (\varepsilon_0 c)^{-1}$ is the impedance of free space. To evaluate a current limit, the smoothed beam radius R_m is re-expressed in terms of the radial aperture as

$$R_m^{\ 2} = \frac{a^2}{f} \quad , \tag{21}$$

where f is a flutter factor that depends on the particle coordinates, but that we have approximated as

$$f = \frac{1 + B\left(\frac{a}{2\pi L}\right)^2}{1 - B\left(\frac{a}{2\pi L}\right)^2} \quad . \tag{22}$$

Equation (20) can be solved for the current I, and Eqs. (4) and (11) can be used to obtain

$$I = \frac{3q\mu}{\pi Z_0 mc^2\beta f^3}\left(\frac{LV}{a}\right)^2 \quad , \tag{23}$$

which expresses the current in terms of the ratio of the voltage to the radial aperture and in terms of μ. It may be convenient to write the current in terms of the zero-current phase advance per focusing period for uncoupled motion. When this is done, we obtain

$$I = \frac{24\mu mc^2\beta^3}{\pi Z_0 qf}\left(\frac{\Gamma a\sigma_0}{L}\right)^2 \quad . \tag{24}$$

As μ approachs 1, Eqs. (23) and (24) give expressions for the peak current, limited by the focusing. Equation (23) is useful when the peak surface electric field limits V/a, and Eq. (24) is useful if the uncoupled phase advance σ_0 is fixed.

The RFQ Transport Equations

The methods, applied above to the rf octupole transport system, also can be applied to the RFQ system. The analogous equations of motion are

$$x'' - B(\sin 2\pi z)x = 0 \quad , \tag{25}$$

and

$$y'' + B(\sin 2\pi z)y = 0 \quad , \tag{26}$$

where

$$B = \frac{qV}{mv^2}\left(\frac{L}{a}\right)^2 . \tag{27}$$

The smooth approximation is

$$x = X(1 - u) \quad , \tag{28}$$

and

$$y = Y(1 + u) \quad , \tag{29}$$

where

$$u = \frac{B}{(2\pi)^2}\sin 2\pi z \quad . \tag{30}$$

The smoothed variables X and Y satisfy

$$X'' + \sigma_0^2 X = 0 \quad , \tag{31}$$

and

$$Y'' + \sigma_0^2 Y = 0 \quad , \tag{32}$$

where σ_0 is the smoothed phase advance per focusing period given by

$$\sigma_0^{\ 2} = \frac{B^2}{8\pi^2} \quad . \tag{33}$$

The space-charge effect is represented by a uniform cylindrically symmetric charge distribution. In this model the smoothed equations of motion, including space charge become

$$X'' + \sigma_0^2(1 - \mu)X = 0 \quad , \tag{34}$$

and

$$Y'' + \sigma_0^2(1 - \mu)Y = 0 \quad , \tag{35}$$

where

$$\sigma_0^2\mu = \frac{qIZ_0 f}{2\pi mc^2\beta^3}\left(\frac{L}{a}\right)^2 \quad . \tag{36}$$

The flutter factor f is given as

$$f = \frac{1 + \frac{B}{(2\pi)^2}}{1 - \frac{B}{(2\pi)^2}} \quad . \tag{37}$$

The quadrupole expressions for beam current, corresponding to Eqs. (23) and (24) for the octupole system, are

$$I = \frac{q\mu}{4\pi z_0 mc^2\beta f}\left(\frac{LV}{a}\right)^2 \quad , \tag{38}$$

and

$$I = \frac{2\pi\mu mc^2\beta^3}{Z_0 qf}\left(\frac{a\sigma_0}{L}\right) \quad . \tag{39}$$

Multiparticle Simulations

Computer simulations were performed to check the analytical predictions and to help understand the particle dynamics. The matched particle distribution (for zero space charge) for the octupole can be calculated, using the "smooth" Hamiltonian.

The smooth coordinates (X, X', Y, Y') are chosen at random and the Hamiltonian is calculated. If the Hamiltonian is less than a specified value, the smooth coordinates are retained and converted into actual phase-space coordinates at a specified time in the rf octupole period. If the resultant coordinates are transformed through many rf periods, the distribution is indeed observed to be matched.

A matched beam that includes space-charge effects is generated by starting with a matched zero-current beam, and gradually increasing the charge assigned to each macroparticle until a specified value is reached. The charge then is held constant, and the beam is transported through many rf periods to obtain a final transmitted current. The initial size of the beam, and the rate at which the current is increased, are adjusted to find a current limit.

We have chosen a set of parameters for evaluation of the octupole transport formulas. We chose 1-MeV protons, a 1-GHz frequency, a 34-kV intervane voltage, and a ratio of radial aperture to focusing period, a/L = 0.1. The zero-current phase advance, calculated from Eq. (13), is 24.6° at the pole tip. A computer run was made with 360 initial particles. Table I summarizes the current limits obtained from the computer simulation and from Eq. (23). The current limit from the formula has been obtained by taking μ = 1.

Table I

A COMPARISON OF CURRENT LIMITS FOR THE OCTUPOLE CHANNEL

I(A) (from formula)	I(A) (computer simulation)
4.0	2.6

The computer simulation value is ∿35% lower than the value predicted by the formula. We believe the most likely explanation for the discrepancy to be either (1) a poor approximation for the flutter factor (Eq. 22), which enters in a sensitive way (as the cube) in Eq. (23), or (2) a possible restriction on the zero-current phase advance, related to instabilities, which might affect the computer simulation.

The RFO Accelerator

An RFO accelerator can be described by the following potential function:

$$U = \frac{V}{2}\left[X\left(\frac{r}{a}\right)^4 \cos 4\Psi + A\, I_0(kr)\cos kz\right]\sin(\omega t + \phi) \quad . \qquad (40)$$

Figure 2 shows the coordinate system, where the coordinates x and z now have dimensions of length. The quantity I_0 is the modified Bessel function and $k = 2\pi/\beta\lambda$. The electric field components are

$$E_r = -\frac{2XV}{a^4} r^3 \cos 4\Psi - \frac{kAV}{2} I_1(kr)\cos kz \quad , \qquad (41)$$

$$E_\Psi = \frac{2XV}{a^4} r^3 \sin 4\Psi \quad , \qquad (42)$$

and

$$E_z = \frac{kAV}{2} I_0(kr)\sin kz \quad , \qquad (43)$$

each multiplied by $\sin(\omega t + \phi)$.

The acceleration efficiency factor A is given by

$$A = \frac{m^4 - 1}{m^4 I_0(ka) + I_0(mka)} \quad , \qquad (44)$$

where m is the vane modulation parameter, shown in Fig. 2. The focusing-efficiency factor X is given by

$$X = 1 - A\, I_0(ka) \quad . \qquad (45)$$

As in the RFQ, the space average longitudinal electric field is proportional to the acceleration efficiency, and is given by

$$E_0 = \frac{2AV}{\beta\lambda} \quad . \qquad (46)$$

We find that for the RFO, there is considerably more accelerating field produced for a given value of m, than for the RFQ.

The pole-tip geometry, which corresponds to the potential function, can be obtained in the same way as for the RFQ.[3] In the middle of the unit cell, where $z = \beta\lambda/4$ (Fig. 2), there is octupole symmetry in the transverse plane. In this symmetry plane all eight pole tips have radius r_0, and their radius of curvature is $r_0/3$, where $r_0 = a/\sqrt[4]{X}$.

Conclusions

We have obtained formulas for an RFO beam transport system and an RFO accelerator and have tested the predictions of a space charge defocusing model against a multiparticle simulation. At high beam currents the simulation shows that the beam will develop a hollow radial distribution, which agrees qualitatively with the model.

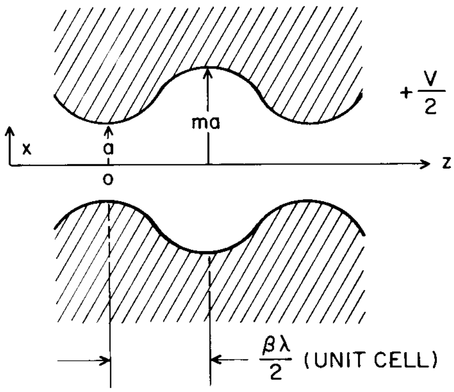

Fig. 2. Radio-frequency octupole pole-tip geometry.

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References

1. R. H. Stokes, T. P. Wangler, and K. R. Crandall, "The Radio-Frequency Quadrupole--A New Linear Accelerator," Proc. 1981 Particle Accelerator Conf., March 11-13, 1981, Washington, DC, IEEE Trans. Nucl. Sci. 28, p. 1999 (1981).

2. H. Bruck, Circular Particle Accelerators, Los Alamos National Laboratory translation (LA-TR-72-10), Chap. 9, p. 91 (1972).

3. K. R. Crandall, R. H. Stokes, and T. P. Wangler, "RF Quadrupole Beam-Dynamics Design Studies," Proc. 10th Linear Accelerator Conf., Montauk, New York, September 10-14, 1979, Brookhaven National Laboratory report BNL-51134, p. 205 (1980).