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ON SPACE CHARGE INSTABILITIES AND EMITTANCE GROWTH IN RF LINACS

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Conditions for minimum transverse emittance growth in rf linacs are discussed. Properties of coherent resonances are estimated analytically. The results are checked by multiparticle simulations for realistic rf linac designs. Most important are the phase dependence of the rf defocusing and the envelope resonance.

Minimum Transverse Emittance Growth

There exist both lower and upper limits for the transverse tune σ if one aims for a high current rf linac design with minimum transverse emittance growth. The lower limit is determined by the phase dependence of the rf defocusing at injection, where the bunch phase width is largest. To avoid transverse emittance growth not only the transverse motion on the average but also that of the bunch head (index h below) must be stable. This condition can be formulated in smooth approximation from the envelope equation:[1]

$$k^2 \approx k_h^2 + k_{rh}^2(1 - \sin\phi / \sin\phi_h) + k_{sh}^2(1 - k/k_h) \quad (1)$$

$k = \varepsilon/a^2$, ε = unnormalized transverse emittance/π
a = average bunch radius
$\sigma = kN\beta\lambda$, $N = 2$ for FD focusing in an Alvarez
$k_{rh}^2 = -q\pi ET\sin\phi_h/(mc^2\lambda\beta^3\gamma^3)$ = rf defocusing
$k_{sh}^2 = 90\Omega qI\lambda M_x/(mc^2\beta^2\gamma^3 a_h^2\, b)$ = space charge effect

It follows that the average tune σ is larger than that at the bunch head σ_h, which in turn is fixed by the condition that the average bunch radius a_h fits well within the aperture. In case the defocusing due to the finite emittance and due to space charge can be neglected compared to the rf defocusing the condition for the minimum average tune results to

$$\sigma_{min}^2 \geq -N^2 q\pi ET\lambda\,(\sin\phi - \sin\phi_h)/(mc^2\beta\gamma^3) \quad (2)$$

Taking into account also a finite emittance would lead to a larger σ_{min}. A high space charge diminishes the influence of the rf phase dependence, allowing a lower tune than required from Eq. 2.

An upper limit for the transverse tune can be derived from the well established requirement to avoid the envelope resonance[2,3], which is assured if the zero current tune σ_0 is below 90°. As σ_0 is obtained by subtracting the space charge effect from σ one obtains for the space charge tune shift in smooth approximation

$$\sigma_{max}^2 \leq \sigma_0^2 - \sigma N180\pi\Omega qI\lambda M_x/(mc^2\beta\gamma^2\, \varepsilon_n\, \Delta\phi) \quad (3)$$

with ε_n = normalized transverse emittance/π,
$\Delta\phi$ = half phase width of the bunch.

The range of transverse tunes at which a linac can be operated with minimum transverse emittance growth gets the smaller the larger λ/β (Eq. 2) and the larger $I\lambda/(\beta\varepsilon_n)$ (Eq. 3) is.
Once one has decided on a tune σ (e.g. according to Eq. 2) the requirement to avoid the envelope resonance transforms into a lower limit for the transverse emittance:

$$\varepsilon_{n,min} \geq N180\pi\Omega qI\lambda M_x\sigma/(mc^2\beta\gamma^2\Delta\phi((\pi/2)^2 - \sigma^2)) \quad (4)$$

Conclusions from Multiparticle Simulations

The results presented above are based on simplifying assumptions, such as modeling the bunch by a uniformly charged ellipsoid and smooth approximation. They were checked by multiparticle simulations based on the Alvarez linac design for the SNQ project[4]. The major linac parameters are protons, frequency 108 MHz, injection energy 450 keV, average electric field on axis E=2MV/m, transit time factor T=0.7 near injection, synchronous phase -35°, bunch phase width at injection about ± 30°, FD quadrupole focusing. The matching parameters at injection and the quadrupole gradients corresponding to a specified tune σ were obtained by an improved version of the CERN ADAPT code[5]. The multiparticle simulations traced 2500 particles fully three dimensional, assuming no symmetries, using an improved version of the CERN MAPRO code[5]. The phase space filling was ellipsoidal, uniform in the four transverse, and independently uniform in the two longitudinal coordinates.

Emittance growth was studied as a function of the transverse emittance. Fig. 1 shows the transverse

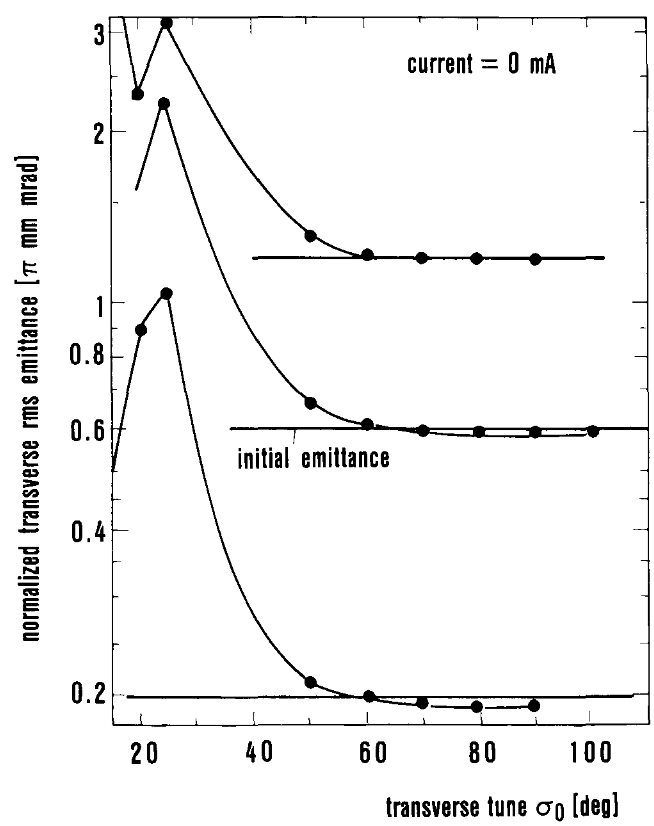

Fig. 1 Transverse emittance growth for zero current

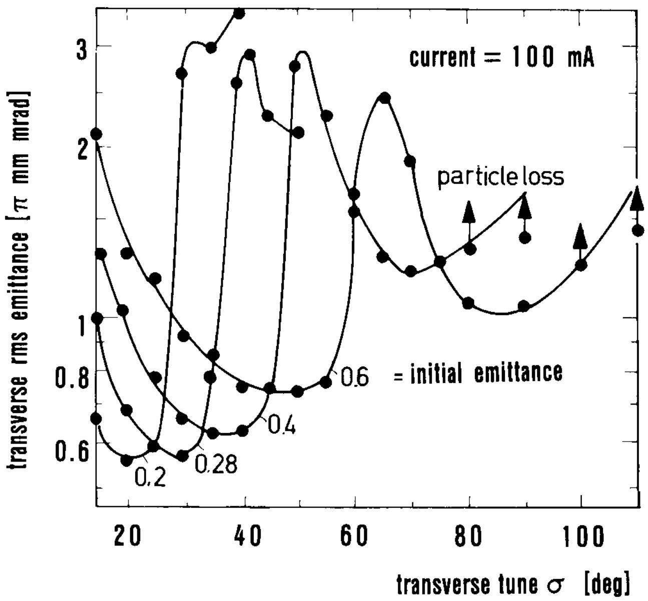

Fig. 2 Transverse emittance growth at 100 mA current

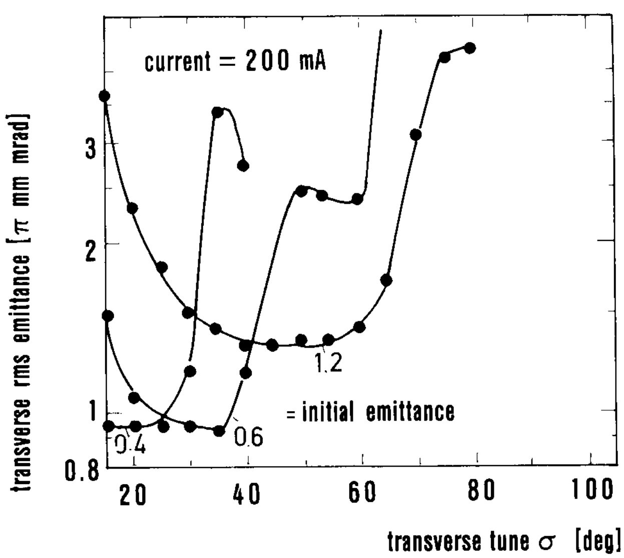

Fig. 3 Transverse emittance growth at 200 mA current

emittance at 14.5 MeV for zero beam current, Fig. 2 for 100 mA, and Fig. 3 for 200 mA. The minimum tune required according to Eq. 2 is 43^{o}, which agrees reasonably well with the results in Fig. 1. This suggests that the large emittance growth observed at lower tunes is indeed caused by the phase dependence of the rf defocusing, also for zero current there is no other cause plausible. With increasing beam current this kind of emittance growth diminishes as can be verified by comparing Fig. 1, 2 and 3. This is in accordance with the fact that the defocusing forces tend to get relatively more dominated by the space charge defocusing which does not depend on the phase for the uniformly charge ellipsoid model. For more realistic phase space density distributions this is no longer the case, then the tune depression due to space charge is largest where the charge density is largest, that is probably near the bunch center.

Analyzing the above figures further most striking is the occurrence of a resonance structure in the emit-

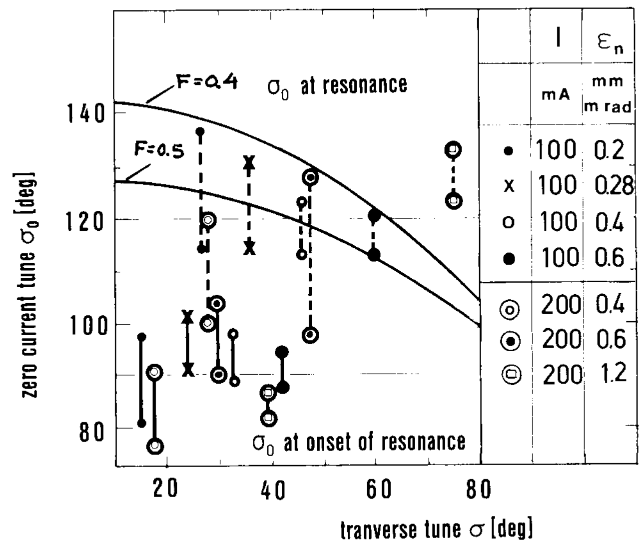

Fig. 4 Transverse tune shift at the maximum, and at the onset of the envelope resonance, over half a betatron oscillation after injection

tance growth once space charge forces come into play. The proof that indeed this is caused by the envelope resonance is given in Fig. 4. Here the upper row of bars is the range over which the zero current tune σ_0 varies in half a betatron oscillation for the tune σ at resonance, analyzed for the cases of Fig. 2, 3. Zero current tunes expected from Eq. 12 for n=2 are also shown. Likewise the lower row of bars is the σ_0 range corresponding to the σ at minimum emittance growth, that is when with increasing σ the emittance growth due to the onset of the resonance starts to balance the decreasing emittance growth due to the rf defocusing. The average σ_0 for this case is about 90^{o} as predicted for the onset of the envelope resonance.

Of further interest is the scaling suggested by Eq.3: the onset of the resonance should scale like IM_x/ε_n, which is verified in Fig. 2 and 3. Increasing the current by a factor of two can transversely be coped with if the transverse emittance is increased by a factor of two as well. Also the growth rates are rather similar in the corresponding cases. Minor differences in this scaling can be contributed to the smaller transverse form factor M_x at the larger emittance, shifting the resonance to slightly higher tunes. The conclusion is that the ratio current/transverse emittance plays a more important role in transverse emittance increase then the closely related brightness[6]. Further, the prediction that at a given tune σ the transverse emittance should not be smaller than indicated by Eq. 4 is verified in Fig. 2 and 3. On the other hand, in the high current cases for a given transverse input emittance a transverse tune can be found at which the transverse emittance growth is minimized. The smaller the input emittance the smaller this optimum tune is, and the more the emittance is blown up due to the phase dependence of the rf defocusing.

Some results for the corresponding longitudinal emittance growth are given in Fig. 5 also as function of the transverse tune. The analysis is complicated by the fact that the matched longitudinal input emittance for non zero current depends both on the current and on the transverse tune. The bunch phase

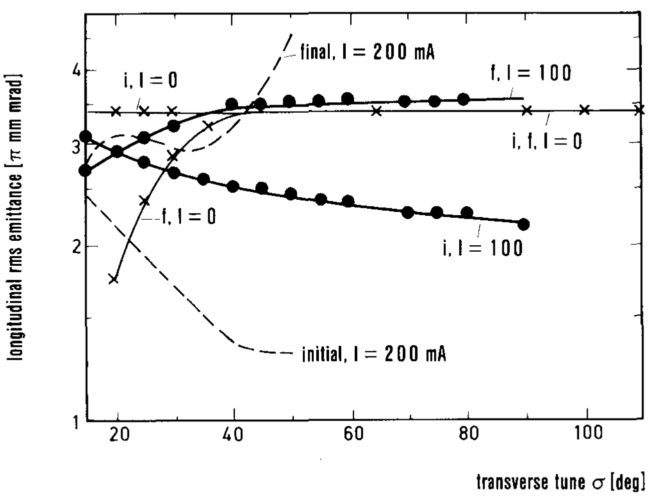

Fig. 5 Longitudinal emittance at 0.45 and at 14.5 MeV for a 0.6 πmm mrad transverse normalized rms emittance at injection

width at injection was chosen to be about ± 30° for all cases, anticipating the rapid growth in longitudinal acceptance due to the acceleration after injection. Thus the corresponding matched energy spread decreases with increasing current and transverse tune. For the zero current case most obvious is the emittance transfer from the longitudinal to the transfer plane at small tunes caused by the phase dependence of the rf defocusing. This effect is hidden at 200 mA current, but is still noticeable at 100 mA, as the final emittance is smaller then the input one at a σ = 15°. At 200 mA and tunes

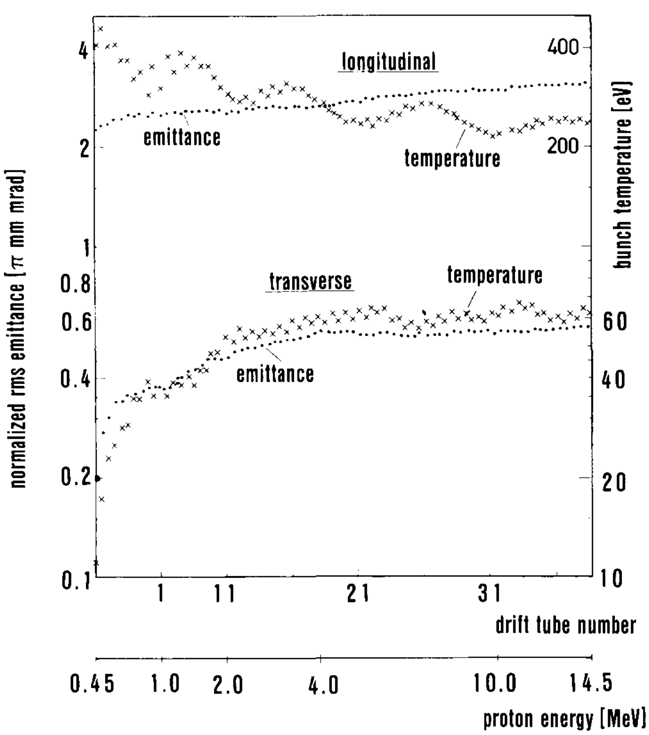

Fig. 6 a Emittance and temperature increase

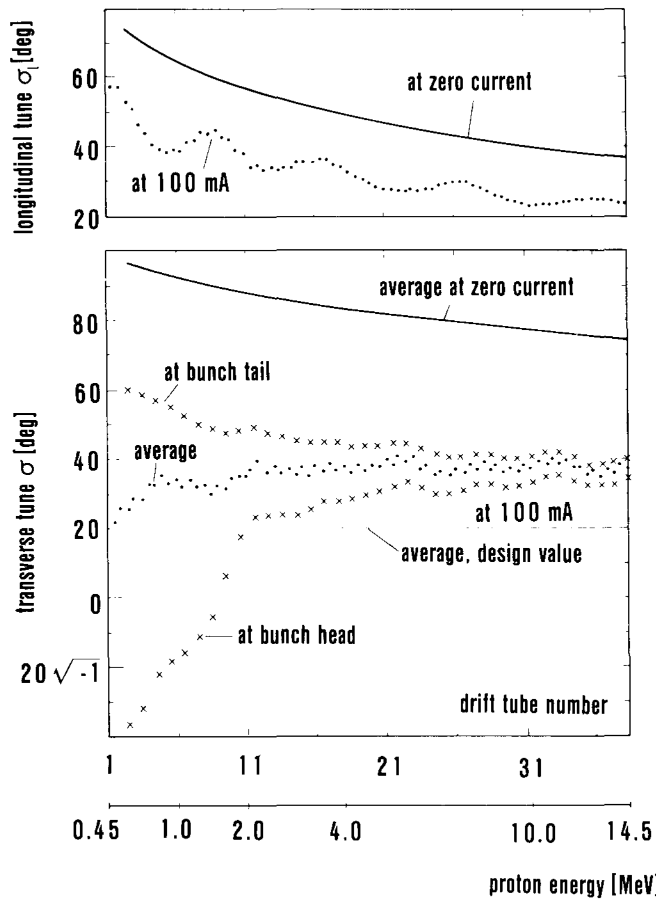

Fig. 6 b Tunes along the linac (100 mA, initial rms emittance 0.2 πmm mrad = $\pi\beta\gamma(\overline{x^2}\,\overline{x'^2}-\overline{xx'}^2)^{1/2}$

above about 40° the longitudinal motion is strongly unstable and the longitudinal emittance growth is large. No resonance structure is detectable in the longitudinal emittance growth.

In Figs. 6 a, b the beam motion is followed along the first tank of the SNQ linac for 100 mA current, a 0.2 πmm mrad normalized transverse rms emittance at injection, and for a constant 20° transverse tune design, which yields according to Fig. 2 a minimum transverse emittance growth for the input emittance chosen. The average tunes at the bunch center are evaluated from the MAPRO results in smooth approximation. The transverse tunes at the bunch head and tail are computed starting from this average tune, taking the tune difference for head and tail from ADAPT. Remarkable is the strong correlation between the transverse tune σ_h at the bunch head and the transverse emittance growth: the growth rate is large as long as σ_h is imaginary, thereafter the emittance still grows but at a much smaller rate. This again supports the view that most of the rapid initial growth is caused by the phase dependence of rf defocusing. Due to this growth the transverse tune gets larger than designed, as for a larger bunch radius the space charge defocusing is reduced. The growth happens nearly evenly in the two transverse coordinates, and the beam matches itself gently to the new conditions. After about 3 MeV energy only very small mismatches remain. The linac design can be further improved by incorporating the theoretically well-understood emittance growth due to

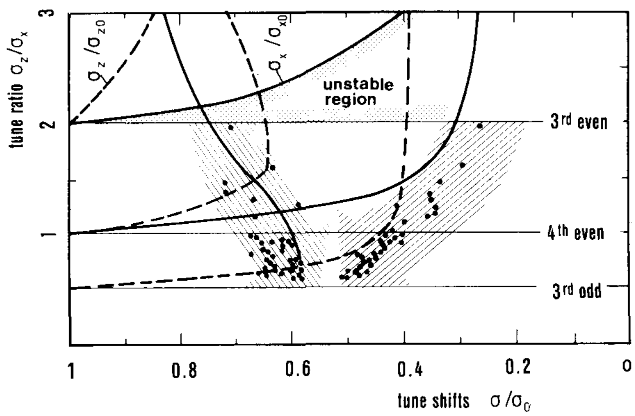

Fig. 7 Stability chart for temperature exchange

the phase dependence of the rf defocusing into a design program like ADAPT[7,8].

The bunch temperatures are evaluated in MAPRO from the rms velocity spreads. They are proportional to the corresponding tune times normalized rms emittance as expected theoretically. Beyond about 4 MeV the emittance growth and the transverse tune have settled and no more temperature or emittance transfer from the longitudinal to the transverse motion is observed, although the longitudinal emittance still is about a factor of 5.5 larger than the transverse one and the corresponding temperature ratio is roughly 4. This suggests that there exist stability regions for the temperature transfer between phase space planes as predicted theoretically[2,9]. The results of the above simulation are plotted in Fig. 7 into a stability chart[9] for an x-y anisotropy ratio $\varepsilon_x/\varepsilon_y = 6$. For about the first 10 linac cells the motion is well inside the unstable region. However, here the linac parameters change rapidly (Fig. 6). Further down the linac the motion is at the edge of the unstable region, where only little temperature transfer is expected theoretically[9].

On Coherent Particle Oscillations

Next the width and the excitation rate of coherent particle oscillations in the bunch are estimated[10]. Courant-Snyders variables $\eta = x/\sqrt{\beta}$ and $\phi = (2\pi/\sigma)\ /\!\int (dz/\beta)$ are used. ϕ is the independent variable which increases by 2π in each transverse focusing period. In this normalized phase plane a n-th order coherent particle oscillation mode is represented by a deformation of the density distribution that varies like

$$F(R)\cos n\,(\Theta + \sigma_n\phi\ /\ 2\pi). \qquad (5)$$

Θ and R are the polar coordinates in this phase plane. If $n\sigma_n = \pi$ the transverse betatron functions have twice the frequency of the mode and can parametrically excite it, because both the coherent space charge and the beams sensitivity to space charge are modulated by the betatron functions.

Although its complete eigenfunction is unknown it can be stated that as a function of ϕ the n-th order mode must fulfill a differential equation of the following form:

$$X'' + (n\sigma/2\pi)^2\ (1 + a_0 + a_1\cos\phi + \dots)\ X = 0 \qquad (6)$$

For $a_0 = a_1 = 0$ this equation describes the unperturbed motion of the azimuthal Fourier components in the η - ϕ - phase plane. σ is the incoherent single particle tune of the unperturbed motion. It is the tune obtained from the zero current σ_0 of the magnet structure and rf defocusing together with the downward shift due to incoherent space charge. Compared to this the coherent space charge has a coherent tune of on the average $n\sigma_n$ arising from the density modulation (Eq. 5). σ_n lies between σ and σ_0 and is represented by the a_0 term. The modulation by the lattice gives $a_1\cos\phi$ and higher terms. Eq. 6 can be rewritten with

$$b_1 = a_1/(1 + a_0) \qquad (7)$$

and

$$\sigma_n^2 = \sigma^2\ (1 + a_0)$$

as

$$x'' + (n\sigma_n/2\pi)^2\ (1 + b_1\cos\phi + \dots)\ X = 0 \qquad (8)$$

Now the modes width and excitation rate are estimated by substituting into Eq. 8 trial solutions

$$X = A\exp(i\phi/2) + \text{complex conjugate}, \qquad (9)$$

neglecting fast terms proportional to $\exp(\pm i3\phi/2)$.

The range of σ_n over which the b_1 term is sufficient to lock the mode onto the lattice is estimated by taking the two special solutions with purely real and imaginary A, and assuming A to be independent of ϕ. The result is

$$\sigma_n = \pi\ /\ (n\sqrt{1 \pm b_1/2}) \qquad (10)$$

To estimate the excitation rate one chooses the tune $n\sigma_n = \pi$ exactly on resonance further allows A to depend on ϕ, neglects A'' compared to A, and considers only the particular solution with $A' = -iA$. One finds

$$A = (1 + i)\ A_0\exp(b_1\phi\ /\ 8) \qquad (11)$$

with a real A_0 and the growth rate $b_1/8$.

The accurate determination of b_1 requires the solution of the radial eigenvalue problem for some specified equilibrium distribution. In the following only a rough estimate is made. Consider first a_0. The coherent force constant for the n-th order mode is obtained by adding to the incoherent force constant at full current a certain fraction F of the incoherent space charge force constant:

$$\sigma_n^2 = \sigma^2 + F\ (\sigma_0^2 - \sigma^2);\ a_0 = \sigma_n^2/\sigma^2 - 1 \qquad (12)$$

For the dipole mode (e.g. bunch center off axis, n = 1, F = 1) the coherent force constant equals the zero current one. With increasing mode order it is reduced towards that of the incoherent motion at full current. For the symmetrical quadrupole mode it is known that F = 0.5, for the independent quadrupole mode corresponding to the envelope resonance discussed above one has n = 2, F = 0.4, and it seems reasonable to estimate that for an independent third order mode n = 3 and F = 0.2.

Next consider a_1. The transverse space charge force in x-direction varies like $M_x/a_x a_y$ in a transverse focusing period in x-x'-coordinates. In the η - ϕ - representation used here this transforms into $s = M_x \beta_x^2 / \sqrt{\beta_x \beta_y}$, since in this plane the effect of a given small extra force constant varies like β_x^2. Requiring that the tune modulation $1 + a_1/a_0 \cos\phi$ equals that of the space charge force constant modulation yields

$$a_1/a_0 = (S - 1)/(S + 1), \; S = s_{max}/s_{min} \qquad (13)$$

The properties of the coherent resonances depend on the actual beam parameters. They are evaluated for a few cases in Table 1.

ε_n	tunes/deg. σ σ_0 σ_2 σ_3	unstable region n = 2	unstable region n = 3	growth rate $b_1/8$ n = 2	growth rate $b_1/8$ n = 3
0.6	65 120 91	$90^{+12}_{-8.4}$		0.054	
0.6	45 91 67 57	90^{+13}_{-9}	$60^{+5.5}_{-4.4}$	0.059	0.041
at injection 0.2	20 97 64 48	90^{+23}_{-13}	$60^{+13}_{-7.7}$	0.091	0.080
after 20 cells 0.2	39 81 59 50	$90^{+12}_{-8.5}$	$60^{+5.3}_{-4.2}$	0.055	0.039

Table 1: Properties of coherent resonances. Current = 100 mA. ε_n = normalized rms emittance at injection in π mm mrad; σ = incoherent tune; σ_0 = zero current tune; σ_2,σ_3 = coherent tunes

For ε_n = 0.6 the evaluation was done at the n = 2 resonance, and at minimum emittance growth (Fig. 2). The predicted bandwidth for the n = 2 resonance agrees qualitatively with the results of Fig. 2. At minimum emittance growth the n = 2 resonance cannot be excited, and σ_3 is just at the edge of the corresponding unstable region. However, no resonances are observed in the tune range σ from 45° to 55° (Fig. 2), so the excitation of the n = 3 resonance must be weak in the simulations.
For ε_n = 0.2 both coherent tunes are outside their unstable regions, so that the observed emittance growth (Fig. 2, 6) must have other causes, as discussed above.

Conclusions

1. To avoid transverse emittance growth due to the phase dependence of the rf defocusing the motion at the bunch head must be stable. The minimum transverse tune required scales as λ/β (Eq. 1, 2). For smaller tunes the transverse emittance grows in a predictable and controllable manner.

2. To safely avoid the envelope resonance (n = 2) the zero current transverse tune must be below 90°, hence there exists a lower limit for the transverse emittance scaling as $I\lambda\sigma / \beta\Delta\phi((\pi/2)^2 - \sigma^2)$ (Eq. 4)

3. In the linac designs presented above temperature exchange between phase planes and the excitation of 3rd order coherent particle oscillations have not been noticable.

4. The width of the 2nd and 3rd order coherent resonances are roughly about 90° ± 10° and 60° ± 5°, there amplitude grows roughly by a factor e in 3, respectively 4 transverse focusing periods.

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Discussion

According to my calculations (for linacs with fixed phase advance with current as designed here), it is a very strong requirement to have the proper emittance area at injection. If you lower the emittance (increase the brightness) too much, the zero-current tune will run into the envelope resonance and you will see growth above the values presented.

Emittance growth from rf defocusing would be seen even if the beam were initially equipartitioned, but I haven't studied the effects separately.