Proceedings of the 1981 Linear Accelerator Conference, Santa Fe, New Mexico, USA

AN ANALYTICAL SOLUTION FOR THE ELECTRICAL PROPERTIES OF A RADIO-FREQUENCY QUADRUPOLE (RFQ) WITH SIMPLE VANES*

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Summary

Although the SUPERFISH program is used for calculating the design parameters of an RFQ structure with complex vanes, an analytical solution for electrical properties of an RFQ with simple vanes provides insight into the parametric behavior of these more complicated resonators.

The fields in an inclined plane wave guide with proper boundary conditions match those in one quadrant of an RFQ. The principle of duality is used to exploit the solutions to a radial transmission line in solving the field equations. Calculated are the frequency equation, frequency sensitivity factors (S), electric field (E), magnetic field (H), stored energy (U), power dissipation (P), and quality factor (Q).

Simple Vanes

For this derivation, simple vanes are vanes whose sides form one straight line from vane tip to vane base, Fig. 1.

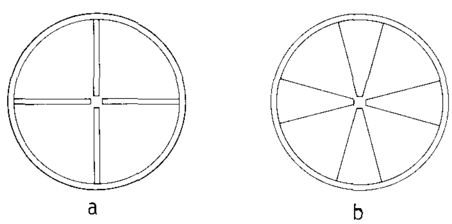
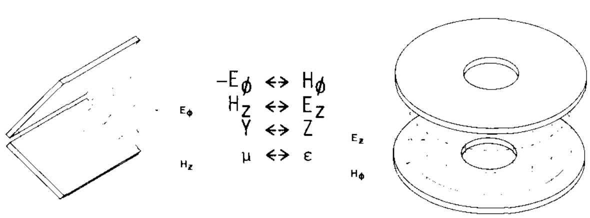


Fig. 1a, b: Examples of Simple Vanes

Principle of Duality

A structure which describes the fields in one resonator of an RFQ is an inclined plane wave guide. The fields in this structure are the duals of those in a radial transmission line.[1] Therefore, all of the well-known equations for radial lines can be used for the inclined plane wave guide with the substitutions shown in Fig. 2.



Inclined Plane   Radial Line

Fig. 2: Replacements by Symmetry

*This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Nuclear Science Division, U.S. Dept. of Energy under Contract No. W-7405-ENG-48.

The field equations then are:

$$E_\phi = \eta\, G_1(kr)\, [A\, e^{j\psi(kr)} - B\, e^{-j\psi(kr)}]$$

$$H_z = G_0(kr)\, [A\, e^{j\theta(kr)} + B\, e^{-j\theta(kr)}]$$

where

$$\eta = \sqrt{\mu/\varepsilon},\quad k = 2\pi/\lambda,$$

$$G_0(kr) = \sqrt{J_0^2(kr) + N_0^2(kr)},$$

$$G_1(kr) = \sqrt{J_1^2(kr) + N_1^2(kr)},$$

$$\theta(kr) = \tan^{-1}[N_0(kr)/J_0(kr)],$$

$$\psi(kr) = \tan^{-1}[J_1(kr)/-N_1(kr)].$$

A and B are constants to be determined. The time varying part of the equations is not shown, and the end effects of the RFQ resonator are not taken into account.

Referring to Fig. 3, the boundary conditions are:

$$E_\phi = E_i \text{ at } r = r_i,$$

$$E_\phi = 0 \text{ at } r = r_L.$$

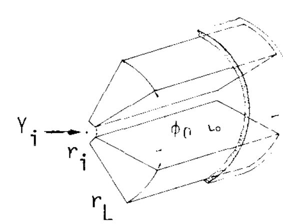


Fig. 3: One RFQ Resonator

Solving for the constants A and B gives for the field equations:

$$E_\phi = E_i\, \frac{G_1}{G_{1i}}\, \frac{\sin(\psi - \psi_L)}{\sin(\psi_i - \psi_L)},$$

$$H_z = (E_i/j\eta)\, \frac{G_0}{G_{1i}}\, \frac{\cos(\theta - \psi_L)}{\sin(\psi_i - \psi_L)},$$

where $G_{1i} = G_1(kr_i)$, $\psi_i = \psi(kr_i)$, $G_1 = G_1(kr)$, etc.; $j = \sqrt{-1}$.

Because the vanes create a highly foreshortened structure, the small argument approximations for the Bessel functions apply. Using these gives for the field equations:

$$E_\phi = E_i \frac{r[1 - (r_L/r)^2]}{r_i[1 - (r_L/r_i)^2]},$$

$$H_z = (E_i/j\eta) \frac{1 + (kr_L)^2\,[\gamma + \ln(kr/2)]/2}{kr_i\,[1 - r_L/r_i)^2]/2}.$$

At resonance, H_z can also be written as:

$$H_z = (E_i/j\eta) \frac{1 - \frac{\gamma + \ln(kr/2)}{\gamma + \ln(kr_i/2)}}{kr_i\,[1 - (r_L/r_i)^2]/2}.$$

Other Parameters

The stored energy in a resonant structure can be expressed as:

$$U = \frac{\varepsilon}{2} \int_V |E|^2\, dv = \frac{\varepsilon}{2} \int_{r_i}^{r_L} |E_\phi|^2\, L_0\, \phi_0\, r\, dr,$$

giving for one resonator:

$$U = \frac{\varepsilon\, E_i^2\, L_0\, \phi_0}{2\, r_i^2\,[1 - (r_L/r_i)^2]^2}[(r_L^4 - r_i^4)/4 - r_L^2(r_L^2 - r_i^2)$$

$$+ r_L^4 \ln(r_L/r_i)];$$

$$U_{TOTAL} = 4U.$$

The power dissipated in an RFQ resonator is composed of two parts, that lost in the vane walls, and that lost in the cylindrical surface. The two parts are given approximately by:

$$P = 2\, R_S/2 \int_{S_{vane}} |J_r|^2\, dS_{vane} + R_S/2\, |J_{r_L}|^2\, L_0 \phi_0 r_L$$

where

$R_S = 1/\sigma\delta$, σ = conductivity, δ = skin depth,

$J_r = H_z(kr)$ = current per unit width.

At resonance,

$$-[\gamma + \ln(kr_i/2)] = 2/(kr_L)^2$$

(see frequency equation below), and P can be written as:

$$P = \frac{R_S\, E_i^2}{2\eta^2} \frac{L_0}{(kr_i)^2\,[1 - (r_L/r_i)^2]^2/4}\ X$$

$$\left\{\left[1 - \frac{\gamma + \ln(kr_L/2)}{\gamma + \ln(kr_i/2)}\right]^2 \phi_0\, r_L\right.$$

$$\left. + \frac{(kr_L)^4}{2}\,[r_L - 2r_i + r_L(\ln(r_L/r_i) - 1)^2]\right\};$$

$$P_{TOTAL} = 4P.$$

The quality factor (Q) for the resonator is defined as:

$$Q = \omega_0 U/P.$$

Substituting for U and P gives:

$$Q = \frac{\varepsilon\, \omega_0\, \phi_0\, k^2\, \eta^2}{4R_S}\ [(r_L^4 - r_i^4)/4 - r_L^2(r_L^2 - r_i^2)$$

$$+ r_L^4 \ln(r_L/r_i)] \div \left\{\left[1 - \frac{\gamma + \ln(kr_L/2)}{\gamma + \ln(kr_i/2)}\right]^2 \phi_0\, r_L\right.$$

$$\left. + \frac{(kr_L^4)}{2}\,[r_L - 2r_i + r_L(\ln(r_L/r_i) - 1)^2]\right\}.$$

The Frequency Equation

The field shapes in an RFQ are greatly distorted from the TE_{210} mode in a conventional cylindrical resonator. This results in a resonant frequency on the order of four times lower than the classical TE_{210} mode.

The input wave admittance, Fig. 2, for an inclined wave guide is given by:

$$Y_i = \left.\frac{H_z}{E_\phi}\right|_i = Y_{oi} \frac{Y_L \cos(\theta_i - \psi_L) + j\, Y_{OL} \sin(\theta_i - \theta_L)}{Y_{OL} \cos(\psi_i - \theta_L) + j\, Y_L \sin(\psi_i - \psi_L)}.$$

Given that the structure, Fig. 3, is resonant, we have:

$Y_L = \infty$, boundary condition,

$Y_i \to 0$, resonant condition,

therefore

$$\theta_i - \psi_L = \pm\pi/2$$

and $\tan(-\theta_i) = \cot \psi_L$, giving for the frequency equation:

$$N_0(kr_i)/J_0(kr_i) = N_1(kr_L)/J_1(kr_L).$$

Using small argument approximations for the Bessel functions gives for the frequency equation:

$$-\ln \frac{kr_i}{2} = \gamma + \frac{2}{(kr_L)^2}$$

where γ = 0.5772... = Euler's constant.

One immediate result is the fact that the resonant frequency does not depend on the angle of inclination of the two planes.

Two examples:

1. RFQ model for Bevatron linac preinjector. (For cross section, see Fig. 1b.)

	Resonant Frequency
r_i = 0.145 cm	SUPERFISH = 372 MHz
r_L = 8.5 cm	Calculation = 370.6 MHz

2. RFQ cold model I for the Numatron Project.[2] (For cross section, see Fig. 1a.)

	Resonant Frequency
r_i = 1.0 cm	SUPERFISH = 453.9 MHz
r_L = 9.5 cm	Calculation = 452.9 MHz

A quick look-up graph for RFQ resonant frequency vs. dimension is given in Fig. 5 for ranges of common interest.

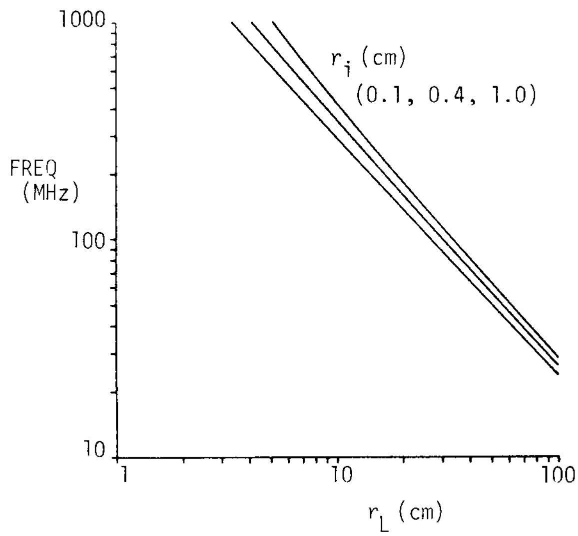


Fig. 5: RFQ Frequency vs. r_L

Sensitivity Factors

The variation in resonator frequency due to dimensional errors can be found by taking differentials of the frequency equation. The sensitivity factors for errors in input and output radius are shown below and plotted in Figs. 6 and 7 for typical radii.

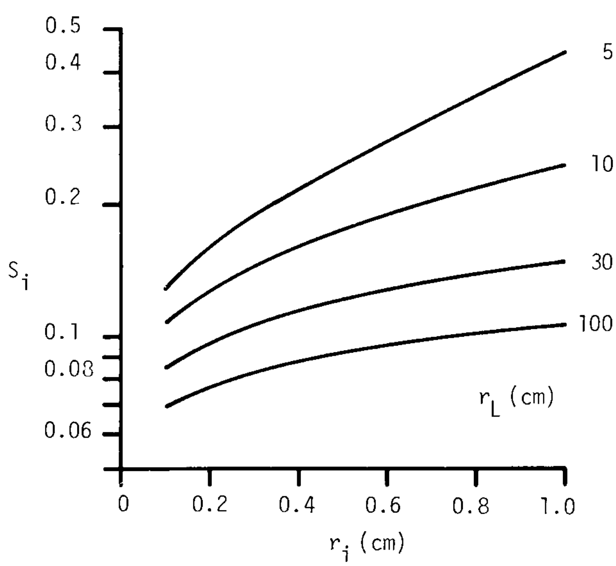


Fig. 6: Sensitivity Factor for r_i

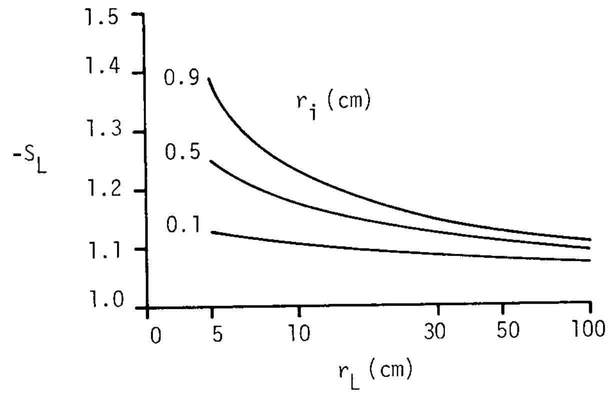


Fig. 7: Sensitivity Factor for r_L

$$S_i = \left.\frac{df/f}{dr_i/r_i}\right|_{r_L = \text{const.}} = [4/(kr_L)^2 - 1]^{-1}$$

$$S_L = \left.\frac{df/f}{dr_L/r_L}\right|_{r_i = \text{const.}} = -\frac{4/(kr_L)^2}{4/(kr_L)^2 - 1}$$

Example:

f = 200 MHz, r_i = 0.31cm, r_L = 15.8 cm, Q = 4000

df = 50 KHz one-half power bandwidth

from figure 6,

$$dr_i = \frac{r_i}{S_i} \times \frac{df}{f} = \underline{6 \times 10^{-4}\ \text{cm}}$$

from figure 7,

$$dr_L = \frac{r_L}{S_L} \times \frac{df}{f} = \underline{3.5 \times 10^{-3}\ \text{cm}}$$

This demonstrates the extreme frequency sensitivity due to dimensional changes.

The sensitivity factor, S_ϕ, for vane tilt errors can be written in terms of the other factors as:

$$S_\phi = \frac{df/f}{d\phi/\phi} = -S_L \frac{\phi}{2} \cot\frac{\phi}{2} .$$

Acknowledgments

The author wishes to thank Dr. K. J. Kim for his help with the frequency equation, and Dr. R. Gough for helpful discussions on electrical parameters and SUPERFISH runs.

References

1 "Fields and Waves in Communication Electronics," Ramo, Whinnery, and Van Duzer, first edition, section 8-14.

2 IEEE Transactions on Nuclear Science, Vol. NS-28, No. 3, (1981) 3510.