BEAM ENVELOPE EQUATION OF LINAC WITH OFF-CENTERED ELLIPTIC EMITTANCE

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Summary

A beam envelope equation of linac with off-centered elliptic emittance has been developed. By means of this equation with given initial conditions the transport of the off-centered beam phase space in various fields can be solved directly.

Introduction

The transverse equation of motion of a single particle in linac is given by the following differential equation

$$a_0(z)\frac{d^2y}{dz^2} + a_1(z)\frac{dy}{dz} + a_2(z)y = F(z), \qquad (1)$$

where y is the transverse coordinate of the particle, z the distance of transport, F(z) the forced term and the coefficients $a_i(z)$ expose the characteristics of the linac, determined by the beam energy, the external electric and magnetic fields, the space charge effect, beam loading and radiation, etc.

Due to construction and installation errors, measuments show that no real phase space in practice is centrosymmetric ellipse, and, therefore, it is of both practical and theoretical interest to develop the beam envelope equation with off-centered elliptic emittance.

Solution of homogeneous equation of motion

First consider the homogeneous equation of a single particle, i.e. the eq.(1) without the forced term F(z),

$$a_0(z)y'' + a_1(z)y' + a_2(z)y = 0, \qquad (1)'$$

where "'" is the differential operator $\frac{d}{dz}$.

Let $y_1(z)$ and $y_2(z)$ be two linearly independent solutions of the eq.(1)', we have the Wronskian[1]

$$\begin{vmatrix} y_1(z) & y_2(z) \\ y_1'(z) & y_2'(z) \end{vmatrix} = \begin{vmatrix} y_1(z_0) & y_2(z_0) \\ y_1'(z_0) & y_2'(z_0) \end{vmatrix} exp\left[-\int_{z_0}^{z}\frac{a_1(z)}{a_0(z)}dz\right], \qquad (2)$$

which is not invariant with respect to the transport variable z because of the term

$$exp\left[-\int_{z_0}^{z}\frac{a_1(z)}{a_0(z)}dz\right].$$

But, by using the relation

$$-\int_{z_0}^{z}\frac{a_1(z)}{a_0(z)}dz = -\int_{\zeta}^{z}\frac{a_1(z)}{a_0(z)}dz + \int_{\zeta}^{z_0}\frac{a_1(z)}{a_0(z)}dz,$$

the eq.(2) can be rewritten as

$$\begin{vmatrix} y_1(z) & y_2(z) \\ y_1'(z)exp\left[\int_{\zeta}^{z}\frac{a_1(z)}{a_0(z)}dz\right] & y_2'(z)exp\left[\int_{\zeta}^{z}\frac{a_1(z)}{a_0(z)}dz\right] \end{vmatrix} = \begin{vmatrix} y_1(z_0) & y_2(z_0) \\ y_1'(z_0)exp\left[\int_{\zeta}^{z_0}\frac{a_1(z)}{a_0(z)}dz\right] & y_2'(z_0)exp\left[\int_{\zeta}^{z_0}\frac{a_1(z)}{a_0(z)}dz\right] \end{vmatrix}, \qquad (2)'$$

which now is invariant with respect to the transport variable z.

Let us find the differential equation of motion which has (2)' as its Wronskian.

Eq.(1)can be rewritten as

$$\frac{d}{dz}\left[y'(z)e^{\int_{\zeta}^{z}\frac{a_1(z)}{a_0(z)}dz}\right] + \frac{a_2(z)}{a_0(z)}e^{\int_{\zeta}^{z}\frac{a_1(z)}{a_0(z)}dz}y(z) = \frac{F(z)}{a_0(z)}e^{\int_{\zeta}^{z}\frac{a_1(z)}{a_1(z)}dz}$$

or

$$\frac{dP(z)}{dz} + a(z)y(z) = G(z), \qquad (3)$$

where y(z) is still the transverse coordinate of the particle, p(z) = m(z)y'(z) is physically interpreted as the equivalent momentum of the particle with its equivalent mass $m(z)=exp\left[\int_{\zeta}^{z}\frac{a_1(z)}{a_0(z)}dz\right]$,

$a(z)=\frac{a_2(z)}{a_0(z)}exp\left[\int_{\zeta}^{z}\frac{a_1(z)}{a_0(z)}dz\right]$ the equivalent linear characteristics of the linac, and $G(z)=\frac{F(z)}{a_0(z)}exp\left[\int_{\zeta}^{z}\frac{a_1(z)}{a_0(z)}dz\right]$ the equivalent forced term.

First consider the homogeneous differential equation of a single particle, i.e. the eq.(3) without the forced term G(z),

$$\frac{dP(z)}{dz} + a(z)y(z) = 0. \qquad (3)'$$

Let $(y_1(z), p_1(z))$ and $(y_2(z), p_2(z))$ be two linearly independent solutions of the eq.(3)', it is not difficulty[2] to show that the very expression (2)' is their Wronskian.

With initial conditions $y_1(0) = 1$, $p_1(0)=0$, $y_2(0)=0$, $p_2(0)=1$ the Wronskian turns out to be a constant 1, i.e.

$$\begin{vmatrix} y_1(z) & y_2(z) \\ P_1(z) & P_2(z) \end{vmatrix} = \begin{vmatrix} y_1(0) & y_2(0) \\ P_1(0) & P_2(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

The general solution of the eq.(3)' is the linear combination of the two linearly independent solutions

$$y(z) = y(0)y_1(z) + P(0)y_2(z),$$

and, by definition, the equivalent momentum

$$P(z) = m(z)y'(z) = y(0)m(z)y_1'(z) + P(0)m(z)y_2'(z) = y(0)P_1(z) + P(0)P_2(z),$$

giving the transformation of the phase space

$$\begin{bmatrix} y(z) \\ P(z) \end{bmatrix} = \begin{bmatrix} y_1(z) & y_2(z) \\ P_1(z) & P_2(z) \end{bmatrix}\begin{bmatrix} y(0) \\ P(0) \end{bmatrix} = R(z)\begin{bmatrix} y(0) \\ P(0) \end{bmatrix} \qquad (4)$$

with the determinant of the transport matrix R(z) equal to unity, i.e.

$$|R(z)| = \begin{vmatrix} y_1(z) & y_2(z) \\ P_1(z) & P_2(z) \end{vmatrix} = 1 \qquad (4)'$$

Consequently, the volume of the beam phase space (y(z), p(z)) is conserved.

Let the initial off-centered elliptic beam phase space be described by the reduced sigma matrix $\sigma_0 = \begin{bmatrix}\beta_0 & -\alpha_0\\ -\alpha_0 & \gamma_0\end{bmatrix}$ with $|\sigma_0| = 1$, or

$$[y_0-\lambda_0 \;\; P_0-\mu_0]\,\sigma_0^{-1}\begin{bmatrix}y_0-\lambda_0\\ P_0-\mu_0\end{bmatrix} = [y_0-\lambda_0 \;\; P_0-\mu_0]\begin{bmatrix}\gamma_0 & \alpha_0\\ \alpha_0 & \beta_0\end{bmatrix}\begin{bmatrix}y_0-\lambda_0\\ P_0-\mu_0\end{bmatrix} = \varepsilon,$$

where (λ_0, μ_0) are the initial coordinates of the ellipse's center, $y_0=y(0)$, $p_0=p(0)$, $\begin{vmatrix}\gamma_0 & \alpha_0\\ \alpha_0 & \beta_0\end{vmatrix} = \begin{vmatrix}\gamma(0) & \alpha(0)\\ \alpha(0) & \beta(0)\end{vmatrix} = 1$, and $\pi\varepsilon$ the area of the ellipse.

The initial elliptic phase space can be rewritten as

$$[y_0-\lambda_0 \;\; P_0-\mu_0]\,\tilde{R}(z)\widetilde{R^{-1}}(z)\begin{bmatrix}\gamma_0 & \alpha_0\\ \alpha_0 & \beta_0\end{bmatrix}R^{-1}(z)R(z)\begin{bmatrix}y_0-\lambda_0\\ P_0-\mu_0\end{bmatrix} = \varepsilon. \tag{5}$$

Combination of (4) and (5) gives the beam phase space at the distance z

$$[y(z)-\lambda(z) \;\; P(z)-\mu(z)]\begin{bmatrix}\gamma(z) & \alpha(z)\\ \alpha(z) & \beta(z)\end{bmatrix}\begin{bmatrix}y(z)-\lambda(z)\\ P(z)-\mu(z)\end{bmatrix} = \varepsilon, \tag{6}$$

with the ellipse's center transformed into

$$\begin{bmatrix}\lambda(z)\\ \mu(z)\end{bmatrix} = R(z)\begin{bmatrix}\lambda_0\\ \mu_0\end{bmatrix}, \tag{6'}$$

and sigma matrix transformed into

$$\sigma^{-1}(z) = \begin{bmatrix}\gamma(z) & \alpha(z)\\ \alpha(z) & \beta(z)\end{bmatrix} = \begin{bmatrix}P_2(z) & -P_1(z)\\ -y_2(z) & y_1(z)\end{bmatrix}\sigma_0^{-1}\begin{bmatrix}P_2(z) & -y_2(z)\\ -P_1(z) & y_1(z)\end{bmatrix}. \tag{7}$$

By the eq.(4)', the determinant of the sigma matrix

$$|\sigma^{-1}(z)| = |\widetilde{R^{-1}}(z)|\,|\sigma_0^{-1}|\,|R^{-1}(z)| = 1,$$

showing that the area of the off-centered elliptic beam phase space is conserved in transport.

The transformation of the elliptic parameters (7) can be rewritten in the following more convenient form

$$\begin{bmatrix}\beta(z)\\ \alpha(z)\\ \gamma(z)\end{bmatrix} = \begin{bmatrix} y_1^2(z) & -2y_1(z)y_2(z) & y_2^2(z)\\ -y_1(z)P_1(z) & y_1(z)P_2(z)+y_2(z)P_1(z) & -y_2(z)P_2(z)\\ P_1^2(z) & -2P_1(z)P_2(z) & P_2^2(z)\end{bmatrix}\begin{bmatrix}\beta_0\\ \alpha_0\\ \gamma_0\end{bmatrix}, \tag{7'}$$

where $\alpha(z) = -\beta'(z)m(z)/2$.

Solution of non-homogeneous equation of motion

Now return to the non-homogeneous differential equation of motion (1).

A particular solution of the eq.(1) is readily found[1] to be

$$\begin{aligned} y_F(z) &= -y_1(z)\Big[\int_0^z \frac{y_2(z)}{a_0(z)}F(z)\exp\Big(\int_3^z \frac{a_1(z)}{a_0(z)}dz\Big)dz - y_F(0)\Big] + \\ &\quad + y_2(z)\Big[\int_0^z \frac{y_1(z)}{a_0(z)}F(z)\exp\Big(\int_3^z \frac{a_1(z)}{a_0(z)}dz\Big)dz + m(0)y_F'(0)\Big] \\ &= \eta(z) + y_1(z)y_F(0) + y_2(z)m(0)y_F'(0), \end{aligned} \tag{8}$$

where the abscissa of the central orbit deviation due to the forced term F(z) is

$$\eta(z) = -y_1(z)\int_0^z \frac{y_2(z)}{a_0(z)}F(z)\exp\Big(\int_3^z \frac{a_1(z)}{a_0(z)}dz\Big)dz + y_2(z)\int_0^z \frac{y_1(z)}{a_0(z)}F(z)\exp\Big(\int_3^z \frac{a_1(z)}{a_0(z)}dz\Big)dz.$$

After differentiation, we have

$$y_F'(z) = \eta'(z) + y_1'(z)y_F(0) + y_2'(z)m(0)y_F'(0)$$

$$\begin{aligned} y_F''(z) &= -y_1''(z)\Big[\int_0^z \frac{y_2(z)}{a_0(z)}F(z)\exp\Big(\int_3^z \frac{a_1(z)}{a_0(z)}dz\Big)dz - y_F(0)\Big] + \\ &\quad + y_2''(z)\Big[\int_0^z \frac{y_1(z)}{a_0(z)}F(z)\exp\Big(\int_3^z \frac{a_1(z)}{a_0(z)}dz\Big)dz + m(0)y_F'(0)\Big] + \frac{F(z)}{a_0(z)} \end{aligned}$$

$$\eta'(z) = -y_1'(z)\int_0^z \frac{y_2(z)}{a_0(z)}F(z)\exp\Big(\int_3^z \frac{a_1(z)}{a_0(z)}dz\Big)dz + y_2'(z)\int_0^z \frac{y_1(z)}{a_0(z)}F(z)\exp\Big(\int_3^z \frac{a_1(z)}{a_0(z)}dz\Big)dz. \tag{8'}$$

Direct substitution of the solution (8) and (8)' into the equation of motion (1) proves that they, indeed, satisfy the non-homogeneous equation

$$a_0(z)y_F''(z) + a_1(z)y_F'(z) + a_2(z)y_F(z) = F(z). \tag{9}$$

Combination of (1) and (9) gives the homogeneous differential equation

$$a_0(z)[y(z)-y_F(z)]'' + a_1(z)[y(z)-y_F(z)]' + a_2(z)[y(z)-y_F(z)] = 0. \tag{10}$$

Therefore, the whole theory for the homogeneous differential equation discussed in the previous section can be resorted to.

By the eq.(4), we get the transformation of the beam phase space

$$\begin{bmatrix}y(z)-y_F(z)\\ P(z)-P_F(z)\end{bmatrix} = R(z)\begin{bmatrix}y_0-y_{F_0}\\ P_0-P_{F_0}\end{bmatrix}, \tag{11}$$

where $y_{f0}=y_f(0)$, $p_{f0}=p_f(0)=m(0)y'_{f0}$.

The solution (8) and (8)' can be rewritten in the matrix form

$$\begin{bmatrix}y_F(z)\\ P_F(z)\end{bmatrix} = R(z)\begin{bmatrix}y_{F0}\\ P_{F0}\end{bmatrix} + \begin{bmatrix}\eta(z)\\ m(z)\eta'(z)\end{bmatrix}, \tag{12}$$

where $\begin{bmatrix}\eta(z)\\ m(z)\eta'(z)\end{bmatrix} = \begin{bmatrix}\eta(z)\\ \theta(z)\end{bmatrix}$ is the central orbit deviation due to the forced term F(z).

Substitution of (12) into (11) gives the foundamental transport formula

$$\begin{bmatrix}y(z)-\eta(z)\\ P(z)-\theta(z)\end{bmatrix} = R(z)\begin{bmatrix}y_0\\ P_0\end{bmatrix}. \tag{13}$$

For the homogeneous case, i.e. F(z)=0, we get $\eta(z)=0$, $\theta(z)=0$ by (8) and (8)', and therefore, the transformation (13) is reduced to (4), showing that $[\eta(z), \theta(z)]$ is, indeed, the central orbit deviation under the action of the forced term F(z).

Substitution of (13) and (6)' into (5) gives the off-centered elliptic beam phase space at the transport distance z

$$[y(z)-(\eta(z)+\lambda(z)) \;\; P(z)-(\theta(z)+\mu(z))]\,\sigma^{-1}(z)\begin{bmatrix}y(z)-(\eta(z)+\lambda(z))\\ P(z)-(\theta(z)+\mu(z))\end{bmatrix} = \varepsilon, \tag{14}$$

where the coordinates of the ellipse's center are $[(\eta(z)+\lambda(z)), (\theta(z)+\mu(z))]$.

For the homogeneous case, i.e. F(z)=0, $\eta(z)=0$, $\theta(z)=0$, the beam phase space (14) is reduced to (6).

According to the reference[3], the projection of the two-dimensional beam phase space (14) onto one-dimensional subspace y(z) gives the upper boundary

$$E_u(z) = [\eta(z)+\lambda(z)] + \sqrt{\varepsilon\beta(z)}, \tag{15}$$

and lower boundary

$$E_l(z) = [\eta(z)+\lambda(z)] - \sqrt{\varepsilon\beta(z)} \qquad (16)$$

of the beam envelope as shown in Fig.1.

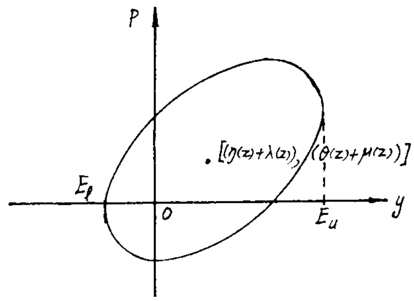

Fig.1 Upper and lower boundary of the beam envelope generated by the off-centered elliptic beam phase space.

Beam envelope equation of linac with off-centered elliptic emittance

By differentiation of (15) and (16) and use of (7)', a simple algebra[2] gives the beam envelope equation of linac with off-centered elliptic emittance

$$a_o(z)[E(z)-(\eta(z)+\lambda(z))]''+a_1(z)[E(z)-(\eta(z)+\lambda(z))]'+$$
$$+a_2(z)[E(z)-(\eta(z)+\lambda(z))] - \frac{a_o(z)\varepsilon^2}{m^2(z)[E(z)-(\eta(z)+\lambda(z))]^3} = 0 \qquad (17)$$

where $\lambda(z)$, the displacement of the ellipse's center due to the initial off-setting (λ_o, μ_o) satisfies the homogeneous differential equation

$$a_o(z)\lambda''(z) + a_1(z)\lambda'(z) + a_2(z)\lambda(z) = 0 \qquad (18)$$

with initial conditions $\lambda(o)=\lambda_o$, $\lambda'(o)=\frac{\mu_o}{m(o)}$,

and $\eta(z)$, the central orbit deviation due to the forced term F(z), satisfies the non-homogeneous differential equation

$$a_o(z)\eta''(z) + a_1(z)\eta'(z) + a_2(z)\eta(z) = F(z) \qquad (19)$$

with initial conditions $\eta(o)=o$, $\eta'(o)=0$

Substitution of eq.(18) and eq.(19) into eq.(17) gives a more concise form of the beam envelope equation

$$a_o(z)E''(z)+a_1(z)E'(z)+a_2(z)E(z) - \frac{a_o(z)\varepsilon^2}{m^2(z)[E(z)-(\eta(z)+\lambda(z))]^3} = F(z) \qquad (20)$$

Given the initial elliptic phase space $\sigma_o = \begin{bmatrix}\beta_o & -\alpha_o \\ -\alpha_o & \gamma_o\end{bmatrix}$

off-centered at (λ_o, μ_o), which by eq.(15) and (16) is translated into the initial beam envelope conditions $\begin{cases} E_o = \lambda_o + \sqrt{\varepsilon\beta_o} \\ E_o' = (\mu_o - \alpha_o\sqrt{\frac{\varepsilon}{\beta_o}})/m(o) \end{cases}$ for the upper boundary and $\begin{cases} E_o = \lambda_o - \sqrt{\varepsilon\beta_o} \\ E_o' = (\mu_o + \alpha_o\sqrt{\frac{\varepsilon}{\beta_o}})/m(o) \end{cases}$ for the lower boundary, we can solve both beam envelope boundaries $E_u(z)$, $E_l(z)$ from the differential equation (20).

The useful complete translation between the initial elliptic phase space and the initial beam envelope conditions is readily proved to be

$$\sigma_o = \begin{bmatrix}\beta_o & -\alpha_o \\ -\alpha_o & \gamma_o\end{bmatrix} = \begin{bmatrix} \frac{(E_o-\lambda_o)^2}{\varepsilon} & \frac{(E_o-\lambda_o)(m(o)E_o'-\mu_o)}{\varepsilon} \\ \frac{(E_o-\lambda_o)(m(o)E_o'-\mu_o)}{\varepsilon} & \frac{\varepsilon^2+[(E_o-\lambda_o)(m(o)E_o'-\mu_o)]^2}{\varepsilon(E_o-\lambda_o)^2} \end{bmatrix}.$$

It is instructive to consider the special case of a single particle with the elliptic phase space contracted to a single point, i.e. $\varepsilon=0$. The beam envelope equation (20) is reduced to the differential equation of motion for a single particle

$$a_o(z)E''(z) + a_1(z)E'(z) + a_2(z)E(z) = F(z),$$

which agrees with the eq.(1).

References

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2. Xie Xi et al., KE JI, 2, 119 (1978).
3. Chen Yinbao, Xie Xi, Chinese Journal of Nuclear Physics, 3, 73 (1981).