EMITTANCE GROWTH

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Abstract

With the growing interest in high-current linear accelerators a broader understanding of emittance growth has become desirable. Recently, the idea of relating emittance growth to the release of electrostatic field energy has been considered as a useful concept for a better understanding of the origin of emittance growth and for deriving simple estimates for growth under certain conditions. The derivation of generalized equations for emittance and field energy valid in periodic focusing and for continuous or bunched beams is outlined. Calculating the field energy for some standard models of charge distribution allows a direct estimate of the emittance growth due to initial mismatch. For the case of emittance transfer between different degrees of freedom due to intrinsic resonances (equipartitioning) we present an approximate formula. These estimates are well confirmed by computer simulation.

I. Introduction

Until recently numerical simulation has been the only tool for calculating emittance growth in linear accelerators or in space charge dominated beam transport. An inherent difficulty with simulation is that it does not provide in a direct way the dependence of emittance growth on different system parameters; furtheron, the origin of emittance growth is often not sufficiently transparent in simulation studies, if a theoretical model is lacked. This is an important issue in design studies, which have maximum current transmission as a goal.

The idea of deriving a differential equation to directly describe the emittance change is not a new one. A first approach has been attempted by Lapostolle[1], who derived a relationship between rms emittance and a space charge field energy term for a continuous beam in a continuous focusing channel. Subsequently, this idea was not pursued further. Of greater practical success have been the rms envelope equations, which where derived by Sacherer[2] in 1D, 2D and 3D. They allowed to calculate envelope oscillations, provided that the rms emittances were known a priori to be constant.

The idea of formulating an equation for the rms emittance revived, after Struckmeier et al.[3] had observed in their numerical simulation work that the extra field energy due to a non-uniform initial charge distribution is transferred into emittance growth within one cell. Postulating the equivalence of field energy and emittance (i.e. thermal energy), they estimated the emittance growth due to homogenization of an initially nonuniform density and found good agreement with simulation. At the Vancouver Particle Accelerator Conference Wangler et al.[4] presented a differential equation for emittance and nonlinear field energy valid for a continuously focused round beam and found that it was accurately observed by simulation. They were thus able to justify rigorously the heuristic formula by Struckmeier et al. and stimulated further work to extend this concept to more general and realistic situations.

In this paper we present a new generalized equation relating emittance to (nonlinear) field energy valid in periodic focusing and full 3D. The equation is tested and confirmed by computer simulation. Its potential to predict initial mismatch emittance growth is obvious: from some preliminary tests it appears to be a valuable tool also for studying longitudinal-transverse emittance transfer. An analytic formulary relevant to this has been derived from the generalized equation and is presented in a different contribution to this conference[5]

II. Generalized Equations for Emittance and Field Energy

We assume single particle equations of motion in x,y,z (deviations from bunch center) with linear and in general time-dependent external focusing forces and arbitrary space-charge forces (nonrelativistic and $s = v \cdot t$)

$$x'' + k_x(s)\, x - \frac{q}{m\gamma^2} E_x\,(x,y,z;s) = 0 \qquad (1)$$

and similar in y,z. The electric field $\underline{E}$ is assumed due to the beam space charge and given by Poisson's equation

$$\nabla\cdot\underline{E} = -\nabla^2\phi = \frac{q}{\varepsilon_o}\, n\,(x,y,z;s) \qquad (2)$$

For constant focusing forces the total energy is a constant and can be written as

$$H \equiv T + V_{ex} + W = \text{const.} \qquad (3)$$

with N the number of particles and

$$T \equiv \frac{1}{2} Nm\gamma^2\,(\overline{x'^2} + \overline{y'^2} + \overline{z'^2}) \qquad (4)$$

(kinetic energy in beam frame)

$$V_{ex} \equiv \frac{1}{2} Nm\gamma^2\,(k_x\,\overline{x^2} + k_y\,\overline{y^2} + k_z\,\overline{z^2}) \qquad (5)$$

(potential energy due to external focusing)

$$W \equiv \frac{\varepsilon_o}{2} \iiint \underline{E}^2\, dx\, dy\, dz \qquad (6)$$

(space charge field energy calculated over a sufficiently large volume V)

where the upper bar indicates mean square averages.

The rms emittances given by

$$\varepsilon_x^2 \equiv 16\,(\overline{x^2}\;\overline{x'^2} - \overline{xx'}^{\,2}) \qquad (7)$$

and similar in y,z are closely related to the respective terms in the "thermal" energy T due to the fact that for a high-current beam $\overline{x^2}$, $\overline{y^2}$ and $\overline{z^2}$ change only little. In principle, any term in T and thus the respective emittance can grow on the expense of the remaining terms in T or the potential and field energy. The energy conser-

vation law Eq. (3) does not limit the exchange of energy among these terms, hence we require an additional constraint involving directly the emittances.

Such an equation can be derived for the general case of periodic, i.e. time dependent focusing where not even the total energy of Eq. (3) is conserved. (For details of the derivation see Ref. 6.) Starting from Vlasov's equation

$$\frac{\partial f}{\partial s} + (\underline{x}' \cdot \underline{\nabla}) f - (\underline{k} - \frac{q}{mv^2} \underline{E}) \cdot \underline{\nabla}_{\underline{x}'} f = 0 \qquad (8)$$

and using the definition of rms emittance in Eq. (7), we can show that

$$\frac{d}{ds} \varepsilon_x^2 = \frac{32q}{mv^2} (\overline{x^2}\ \overline{x'E_x} - \overline{xx'}\ \overline{xE_x}) \qquad (9)$$

and similar for y,z. Here the averages are taken with the distribution function as weight. The $\overline{x'^2}$ (as well as $\overline{y'^2}$, $\overline{z'^2}$) has been eliminated after multiplying Eq. (8) by x^2, xx', x'^2 etc. and obtaining the respective moment equations.

Eq. (9) is not a useful equation, since the averages involving the electric field are yet unknown. By adding to Eq. (9) its counterparts in y, z, we can show[6] that the electric field averages can be expressed in terms of the electric field energy

$$\frac{1}{\overline{x^2}} \frac{d}{ds} \varepsilon_x^2 + \frac{1}{\overline{y^2}} \frac{d}{ds} \varepsilon_y^2 + \frac{1}{\overline{z^2}} \frac{d}{ds} \varepsilon_z^2 = -\frac{32}{mv^2 N} \left(\frac{dW}{ds} - \frac{dW_u}{ds}\right) \qquad (10)$$

In Eq. (10) we have emitted a bunch shape factor λ_3 in front of dW_u/ds, which is in practice very close to unity. (For a Gaussian density it is 1.051, and for a parabolic 1.006.) Also, we have assumed rotational symmetry in deriving Eq. (10), i.e. $\overline{x^2} = \overline{y^2}$ and $\varepsilon_x = \varepsilon_y$. W_u is defined as electric field energy of the equivalent (i.e. identical rms size) uniformly filled ellipsoid. This uniform field energy calculated inside a large sphere of radius R results as:

$$W_u = \frac{N^2 q^2}{40\pi\varepsilon_o} \left[\frac{9}{c} (1 - f + \frac{c^2}{a^2} f) - \frac{5}{R} \right] \qquad (11)$$

where a is the semi-axis in x, y, and c in z. f(c/a) is the well-known geometry factor[7] which can be approximated by $(3c/a)^{-1}$ as long as $0.8 \lesssim c/a \lesssim 5$.

The significance of Eq. (10) is that its r.h.s. vanishes for a uniform charge distribution; the change of total emittance is thus related to the change of "nonlinear" field energy, i.e. the difference between actual field energy and that of a uniform bunch (where E increases linearly from the bunch center). This "nonlinear" field energy is thus responsible for emittance growth and we want to examine to what extend we can use Eq. (10) to estimate possible emittance growth in real situations.

III. General Properties

1. Minimum Field Energy Theorem

We consider the variational expression

$$S \equiv W + \alpha_1 \overline{x^2} + \alpha_2 \overline{y^2} + \alpha_3 \overline{z^2} , \qquad (12)$$

with α_i Lagrange multipliers to keep the rms dimensions constant. To find the minimum field energy density distribution we calculate the variation of S under a density perturbation δn and find, after partial integration,

$$\delta S = \iiint_V \left| \phi + N^{-1} (\alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2) \right| \delta n \, dxdydz = 0 \qquad (13)$$

with a boundary integral negligible for a large enough integration volume V. δn is defined by an arbitrary displacement vector $\delta\underline{x}$, hence $\delta n = \nabla n \cdot \delta\underline{x}$. We thus find that $\delta S = 0$ requires either $\phi = -N^{-1}(\alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2)$ (interior of beam) or $n = 0$ (exterior), which corresponds to a uniformly charged ellipsoid as minimum field energy solution.

It is well-known from computer simulation that a high-current beam has a tendency to approach a uniform density with rounded off edge in the space charge limit. For a nonuniform starting density we thus expect the excess "nonlinear" field energy to be transferred into emittance. A uniform starting distribution, on the other hand, requires some small amount of nonlinear field energy to develop the rounded off edge. In this case its rms emittance could even shrink by some small amount, which is not forbidden by Liouville. This has been shown previously[6] for a 2d beam with initial KV-distribution. here it is useful to note the difference between rms emittance and real emittance: a beam with real emittance zero going through a nonlinear lens will have a finite rms emittance behind it due to a distortion of the zero area emittance figure. This can be reversed by a second lens of opposite sign.

2. Shielding

For a spherical bunch of radius a with Gaussian distribution function we can show, by expanding the density in Eq. (2) to its leading term that (except for $r/a \approx 0$)

$$n = n(0) \left[1 - \frac{e^{(r-a)/\lambda_D}}{r/a} \right] \qquad (14)$$

where the Debye or screening length λ_D is given by

$$\lambda_D^2 = \frac{\overline{x'^2}}{\omega_p^2} \qquad (15)$$

and the plasma frequency by

$$\omega_p^2 = \frac{q^2 n(0)}{\varepsilon_o m v^2} \qquad (16)$$

In the harmonic betatron oscillation approximation one finds readily that

$$\frac{\lambda_D}{a} \approx \frac{1}{\sqrt{15}} \frac{\nu}{\nu_o} \qquad (17)$$

(a similar result with $1/\sqrt{15}$ replaced by 1/3 is found for a waterbag distribution).

In the high-current limit we thus obtain that self-consistent distributions are uniform in space, except for a boundary sheath of thickness λ_D.

IV. Comparison with Computer Simulation

By inspecting Eq. (10) we can, in principle, identify the following sources of emittance growth:

Initial Mismatch - Continuous Focusing

An initially nonuniform density profile deviating strongly from the self-consistent profile described by Eq. (14), (17) leads to emittance growth, which can be directly integrated from Eq. (10) for a spherical bunch, with $k_x = k_y = k_z$ and $\varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon$:

$$\frac{\Delta\varepsilon^2}{\varepsilon^2} \approx \frac{1}{3}\left(\frac{\nu_o^2}{\nu^2} - 1\right) \Delta U \qquad (18)$$

For the integration we have used that the rms size remains practically constant and defined the "nonlinear" field energy

$$U \equiv \frac{W - W_u}{W_1} \qquad (19)$$

with a normalization constant $W_1 \equiv N^2q^2/(40\pi\varepsilon_o a)$, which is the field energy of a uniform sphere of the same rms radius (= a). From the initial and final values of U we calculate the ratio of emittances

$$\frac{\varepsilon_f}{\varepsilon_i} = \left[1 - \frac{1}{3}\left(\frac{\nu_o^2}{\nu^2} - 1\right)(U_f - U_i)\right]^{1/2} \qquad (20)$$

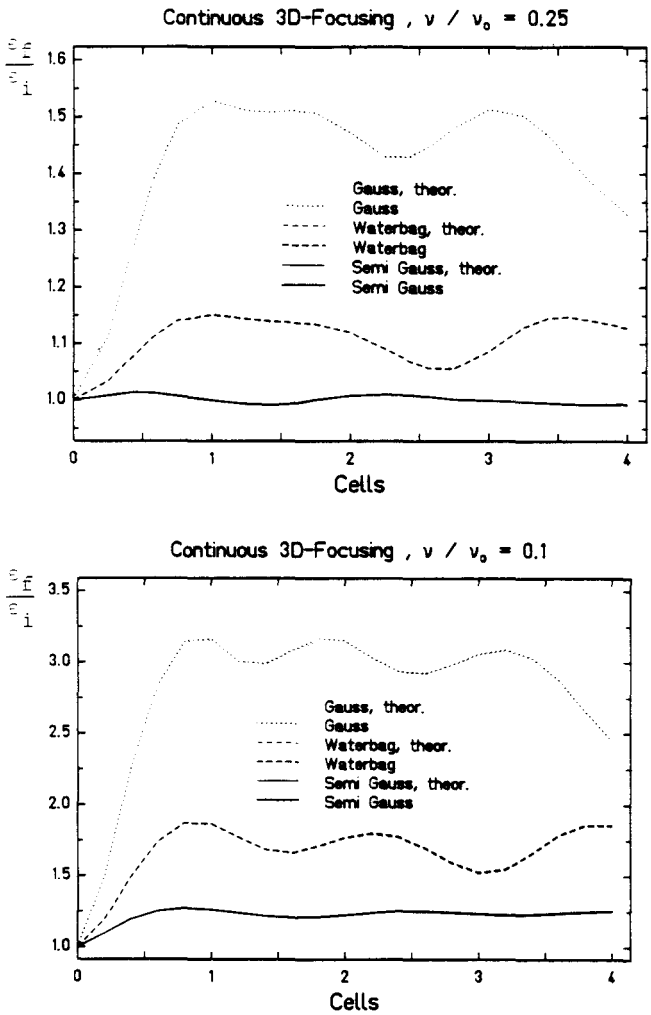

Fig. 1: Computer simulation of spherical bunch with continuous focusing. Emittance growth factors from Eq. (20) and through direct evaluation practically coincide.

This agrees with the respective formula in 2d beams, except for 1/3 replaced by 1/2.

In Fig. 1 we show results of simulation with different initial distributions matched to a constant focusing channel by means of the rms envelope equations[2]. We have calculated U as a function of time in the following way: W is determined as Coulomb potential summed over all pairs of particles, W_u according to Eq. (11). We have then compared $\varepsilon_f/\varepsilon_i$ according to Eq. (20) with the direct result from scatter plots. The agreement in runs with 5000 particles is excellent; deviations are typically one per cent only.

Significant emittance growth occurs only if U_i is noticeably larger than the self-consistent final value. For practical purposes we can thus ignore U_f in Eq. (20) and obtain

$$\frac{\varepsilon_f}{\varepsilon_i} \lesssim \left[1 + \frac{1}{3}\left(\frac{\nu_o^2}{\nu^2} - 1\right) U_i\right]^{1/2} \qquad (21)$$

For the Gaussian case in Fig. 1 we have $U_i = 0.26$ and thus from Eq. (21) $\varepsilon_f/\varepsilon_i \lesssim 1.52$ (3.1) for $\nu/\nu_o = 0.25$ (0.1). For the waterbag distribution the respective values are $U_i = 0.057$ and thus $\varepsilon_f/\varepsilon_i \lesssim 1.13$ (1.7). These values agree pretty well with the average values of Fig. 1.

It will be noted that the semi-Gaussian distribution with uniform initial density behaves different from what we have predicted in section III.1: the emittance shows a slight initial increase rather than a decrease. This is due to the fact that microscopically our distribution is set up by random numbers. Hence, there is an extra electrostatic field energy as compared with a strictly uniform distribution, which leads to an equivalent emittance growth.

Initial Mismatch - Periodic Focusing

The validity of Eq. (10) in periodic focusing with non-spherical bunches needs to be examined separately. This is due to the fact that we have omitted the bunch-shape factor λ_3, which is exactly justified only for a uniform bunch. In addition, the envelopes in Eq. (10) vary periodically and it is not straightforward how to integrate the l.h.s. of Eq. (10). We have integrated Eq. (10) formally by ignoring the envelope variations:

$$\frac{\Delta\varepsilon_x^2}{\overline{x^2}} + \frac{\Delta\varepsilon_y^2}{\overline{y^2}} + \frac{\Delta\varepsilon_z^2}{\overline{z^2}} = -\frac{32}{mv^2N} \Delta (W - W_u) \qquad (22)$$

For a bunch with a roughly 2:1 ratio of longitudinal to transverse semi-axi and transverse periodic solenoid focusing we compare in Fig. 2a the l.h.s. and the r.h.s. of Eq. (22) and find good agreement in the average. We have also plotted in Fig. 2a the individual l.h.s. terms of Eq. (22) and find that they are not identical. We note that in this case we have chosen $\overline{x'^2}$, $\overline{y'^2}$ and $\overline{z'^2}$ equal as initial condition, hence equal thermal energies. (According to Eq. (6) we have $\overline{x'^2} = \varepsilon^2/16\,\overline{x^2}$ for upright phase space ellipses in x, x' and similar in y, y' and z, z'). The initial mismatch field energy is transferred into roughly equal increments of thermal energies in all three directions, but at later times the longitudinal thermal energy deviates from the transverse ones.

In Fig. 2b we have plotted the numerical emittance growth factors and compared them with the respective theoretical values. Here we have determined "theoretical" growth factors by distributing formally the total

nonlinear field energy in equal parts into the three l.h.s. terms of Eq. (10). This "equipartitioning" hypothesis is not strictly observed by the simulation, as observed above, although the deviation is not too large. In fact, we see from Fig. 2 that "equipartitioning" is re-established at repeated times.

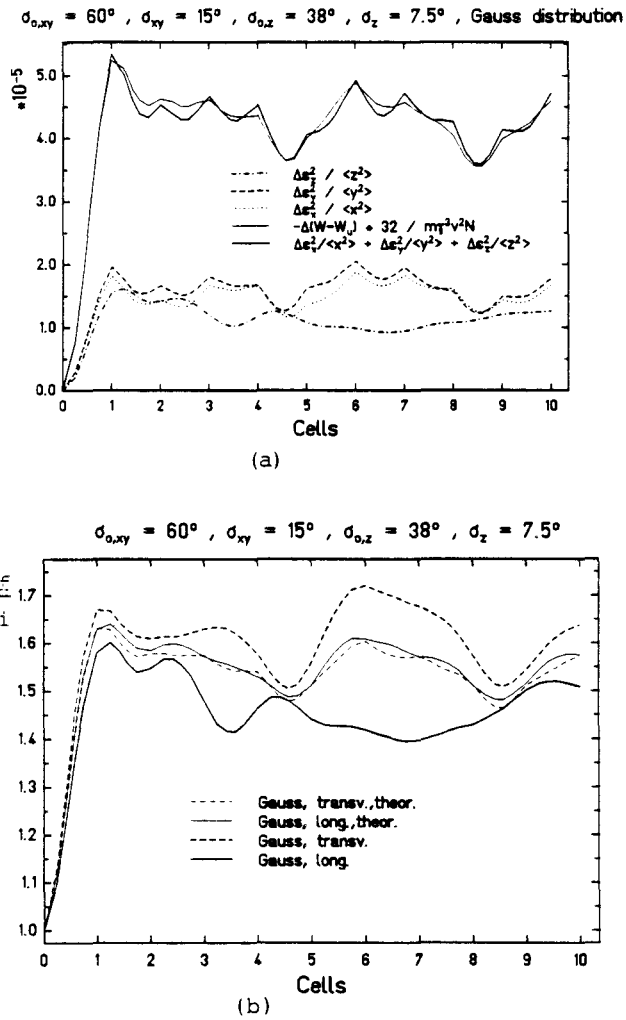

Fig. 2a, b: Non-spherical initial Gaussian density bunch with periodic (interrupted solenoid) focusing and initial equipartitioning ($\varepsilon_z = 2\varepsilon_{x,y}$; $\sigma_z = 1/2\ \sigma_{x,y}$) confirming relationship between total emittance change and nonlinear field energy change (a). Emittance growth factors are compared with equipartitioned theoretical values (b).

Initial Mismatch and Emittance Transfer - Periodic Focusing

In Fig. 3 we show results for a case with $\sigma_{o,z}$ and σ_z different from Fig. 2, causing a strong initial deviation from equipartitioned. The ratio of thermal energies longitudinal: transverse is chosen 7.5:1, and from previous work on equipartitioning and emittance transfer[8,9] we expect an exchange of thermal energies rsp. emittances. In Fig. 3 we actually distinguish two separate mechanisms:

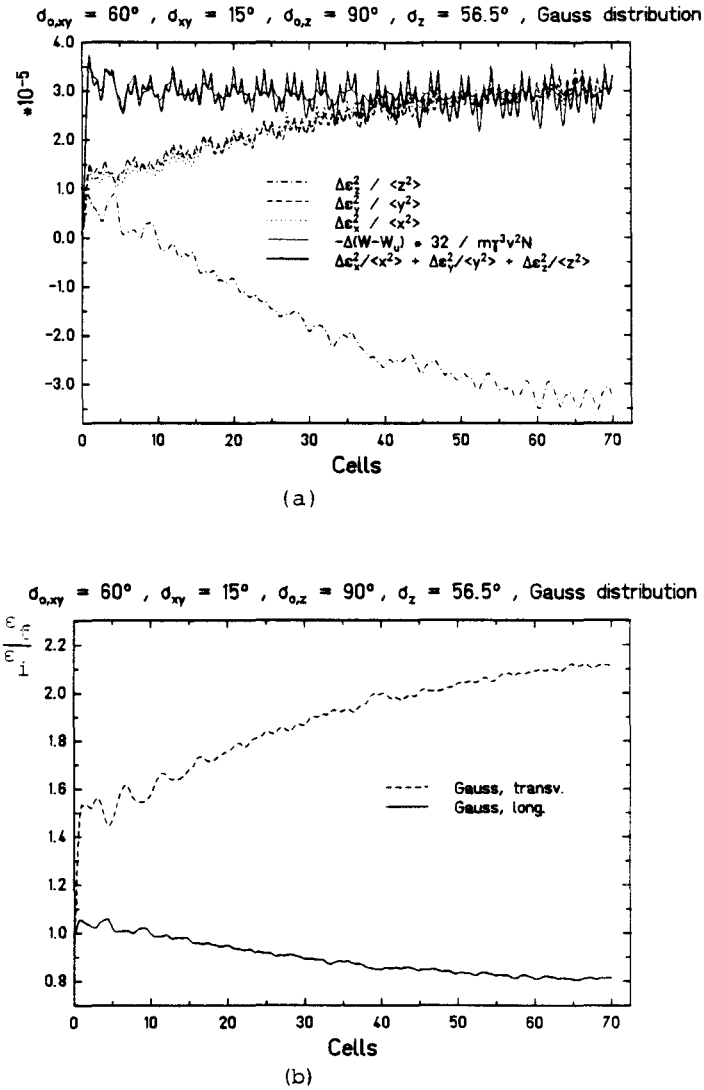

Fig. 3a, b: Same as in Fig. 2, except for $\sigma_z = 3.77\ \sigma_{x,y}$ to achieve initial non-equipartitioned (7.5:1) bunch. Relationship Eq. (20) is again confirmed (a), after first cell also Eq. (23). Emittance growth factors show slow approach to equipartitioning (b).

1. A rapid decrease - within the first cell - of nonlinear field energy (initial mismatch), which is predominantly transferred into the x and y terms of Eq. (10), i.e. the directions with smaller initial thermal energy.

2. After one or two coherent oscillations there is a slow exchange of transverse with longitudinal thermal energies. Besides a small coherent oscillation the nonlinear field energy is practically constant, hence we can use the following relationship

$$\frac{\Delta\varepsilon_x^2}{\overline{x^2}} + \frac{\Delta\varepsilon_y^2}{\overline{y^2}} + \frac{\Delta\varepsilon_z^2}{\overline{z^2}} \approx 0 \qquad (23)$$

in agreement with Fig. 3a.

Eq. (23) establishes the approximate invariance of total thermal energy during the slow emittance transfer process. After 50 cells the initial 7.5:1 imbalance of thermal energies has been reduced to nearly 1.5:1 and practically levelled off. The respective emittance growth factors are shown in Fig. 3b.

In 2D continuously focused beams the phenomenon of emittance transfer has been studied more systematically in a paper presented at this conference[1] .

Eq. (23) can be used to calculate approximately the final emittance growth, if as a second condition we assume that the final thermal energies are equipartitioned and the rms envelopes remain practically constant. With these conditions and $\varepsilon_x = \varepsilon_y = \varepsilon_\perp$ we solve for

$$\frac{\varepsilon_{\perp,f}}{\varepsilon_{\perp,i}} = \left(\frac{1}{3} + \frac{2}{3} \frac{\varepsilon^2_{\perp,i}}{\varepsilon^2_{z,i}} \frac{\overline{z^2}}{\overline{x^2}} \right)^{1/2} \qquad (24)$$

and

$$\frac{\varepsilon_{z,f}}{\varepsilon_{z,i}} = \left(\frac{2}{3} + \frac{1}{3} \frac{\varepsilon^2_{z,i}}{\varepsilon^2_{\perp,i}} \frac{\overline{x^2}}{\overline{z^2}} \right)^{1/2} \qquad (25)$$

where f indicates final and i initial, after the initital mismatch emittance growth of the first cell has been estimated separately. Here we use from Fig. 3a that $\varepsilon_{\perp,i}/\varepsilon_\perp \approx 1.5$ and $\varepsilon_{z,i}\ \varepsilon_z \approx 1.05$ and obtain for the final total emittance growth factors $\varepsilon_{\perp,f}/\varepsilon_\perp \approx 2.06$ and $\varepsilon_{z,f}/\varepsilon_z \approx 0.76$, which agrees within 5 % with the final values in Fig. 3a. We observe that the initial mismatch emittance growth factors can be also estimated from Eq. (10), if we postulate that the nonlinear field energy only goes into the directions that have had initially much less thermal energy (here x,y). Further formulary relevant to this is found in Ref. 5 and 6.

Periodic Focusing - Structure Resonances

A further well-known source of emittance growth are structure resonances, for instance by choosing $\sigma_o > 90°$. Multiplying Eq. (10) by $\overline{x^2}$, which varies periodically with the focusing structure, we anticipate the possibility of emittance growth, if the nonlinear field energy term has the same frequency content as the $\overline{x^2}$ due to an appropriate multipole oscillation of beam density. It is known that $\sigma_o = 60°$ is a safe way to avoid this source.

Statistical Effects

The early emittance growth of the semi-Gauss distribution in Fig. 1 has been explained by the random initial particle coordinates. This is not a problem in the real world, but for numerical simulation with relatively few particles and a direct particle to particle force calculation. Hence some care is necessary for small tune depression (i.e. dominant space charge) as shown in the following example: The case of Fig. 1 with $\nu/\nu_o = 0.1$ has given 27 % of emittance growth due to this mechanism for 5000 particles and 135 % for 500 particles. For $\nu/\nu_o = 0.25$ the respecitve numbers are 1 % and 14 %. These numbers can be substantially reduced by avoiding random numbers and choosing regular initial distances between the particles.

Conclusion

We have shown that a generalized equation for emittance and field energy is well confirmed by simulation in 3D and periodic focusing (interrupted solenoid). Emittance change can occur due to rapid initial mismatch and "slow" energy transfer for initially nonequipartitioned bunches. The generalized equation has been used to derive simple estimates for the emittance change. In a further contribution to this conference[5] more specific formulae have been presented. A more systematic comparison with simulation will be a further step.

Acknowledgment

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