

# IMPEDANCE OF CAVITIES WITH BEAM PORTS ABOVE CUT-OFF

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## ABSTRACT

Accelerating cavities made out of perfect conducting metal have only a finite number of resonant modes due to the (necessary) beam ports. Above the lowest cut-off frequency of the travelling modes in the beam pipe no undamped resonance can exist. Thus the impedance in this range of frequency cannot be obtained by resonator codes such as URMEL [1]. The continuous impedance above cutoff for cavities of arbitrary shape but cylindrical symmetry can be calculated by exciting the cavity with a beam current equal to the Fourier transform of a point charge traversing the structure. Solving Maxwell's equations for the fields yields the longitudinal electric field and thus by integration the impedance. URMEL has been extended to enable such calculations by setting up an inhomogeneous, complex matrix equation for a given frequency. Solving for that complex system of linear equations yields the electromagnetic field and thus the complex impedance.

## INTRODUCTION

Some important aspects in the design of accelerating components are their influence on the beam, i.e. the instabilities as mode coupling or the power loss, and their own behaviour in the working machine, e.g. heating of walls, etc.. The calculation of the impedance and the evaluation of the wake potential are two equivalent ways to study these questions. In this paper a numerical method will be worked out to calculate the impedance for a cylindrically symmetric structure, as, for example, a cavity with beam ports.

Any non-smooth parts in the accelerator geometry (e.g. cavities, vacuum chamber joints, etc.) cause so-called wake fields which are excited by the beam of charged particles. To estimate these effects it is sufficient to integrate the wake fields over the time needed to pass the structure. This gives the wake potential which is a function of the position. The impedance describes the wake force in the frequency domain and can be obtained from the wake potential by Fourier transformation.

For a closed cavity the impedance as a function of frequency has only discrete  $\delta$ -function-like peaks (infinite in number) located at the resonant frequencies which are the only modes of the wake fields that can be excited in the cavity. There are a number of computer codes to evaluate the lowest resonant frequencies and quality factors for cavities needed for the impedance calculation, e.g. [1].

Cavities as components of an accelerator are imbedded in the vacuum chamber. Above the lowest cut-off frequency for travelling waves in this beam port no ideal resonance can exist in the cavity since fields may travel out of the cavity into the beam ports. The nonzero values of the impedance form a continuous spectrum above the cut-off frequency. Consequently calculating the impedance by using resonator codes can not be used. A way to calculate the impedance as a function of frequency including the region above cut-off will be explained in this paper resulting in the extended program URMEL-I. Since the work is still in progress only preliminary results are presented.

## DEFINITION OF THE IMPEDANCE

The definition of the impedance for a resonant cavity is deduced from the representation of a cavity as a RLC network: This network consists of a sequence of resonators, so that the impedance of the cavity is given by the summation over the impedances of the resonators. The impedance of one resonator is given by :

$$Z(\omega) = \frac{1}{i\omega C + 1/i\omega L + 1/R} \quad (1)$$

Since it is intended to calculate the impedance below cut-off using the data given by a resonator code such as URMEL[1], the above formula will be expressed now in terms of the frequency  $\omega$ , the resonant frequency  $\omega_0$  and the loss parameter  $k_0$ , i.e.  $L = 2k_0/\omega_0^2$ ;  $C = 1/2k_0$ ;  $R/Q = \omega_0 L$  is used.  $1/Q$  expresses the losses. As the resonator code assumes perfect conducting metal as walls,  $1/Q$  is zero for the solution so that the impedance for any frequency below cut-off can be calculated by [2] :

$$Z(\omega) = \sum_j \frac{2i\omega k_j}{\omega_j^2 - \omega^2}, \quad (2)$$

with the resonant frequencies  $\omega_j$  and the loss parameters  $k_j$ ,  $j=1,2,\dots$

In general the impedance can be defined as the Fourier transform of the wake potential, as already mentioned. Exciting the cavity structure by a current on the axis allows the calculation of the longitudinal impedance in dependence of a chosen frequency:

A point charge on the axis of a cylindrically symmetric structure has the charge density :

$$\rho(r, \varphi, z, t) = Q\delta(r)\delta(z - vt), \quad (3)$$

and its current density equals

$$\vec{j}(r, \varphi, z, t) = \rho(r, \varphi, z, t) \cdot \vec{v} = Qv\delta(r)\delta(z - vt)\vec{e}_z, \quad (4)$$

with  $Q$  = charge and  $v$  = speed of the point charge.

This point charge produces the longitudinal wake potential seen by a particle following in time an amount  $\tau$

$$w(\tau) = \frac{1}{Q} \int_{-\infty}^{\infty} E_x(r, \varphi, z = vt - v\tau, t) v dt, \quad (5)$$

which on the other hand is [3] :

$$w(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau} Z(\omega) d\omega. \quad (6)$$

Some transformations finally lead to the equation for the longitudinal impedance :

$$Z(\omega) = \frac{1}{Q} \int_{-\infty}^{\infty} \vec{E}_x(r = 0, \varphi, z, \omega) e^{i(\omega/v)z} dz, \quad (7)$$

where  $\vec{E}_x$  is the Fourier transform of the longitudinal electric field  $E_x$ .

In concluding this section it should be stressed that there is an obvious duality between the study of wake force effects in the time and in the frequency domains:

TIME DOMAIN	FREQUENCY DOMAIN
charge density	transformed charge density
$\rho = Q\delta(r)\delta(z - vt)$	$\tilde{\rho} = \frac{Q}{v} e^{-i(\omega/v)z} \delta(r)$
current density	transformed current density
$\vec{j} = Qv\delta(r)\delta(z - vt)\vec{e}_z$	$\vec{\tilde{j}} = Q e^{-i(\omega/v)z} \delta(r)\vec{e}_z$
wake potential	impedance
$W(s) = \frac{1}{Q} \int_{-\infty}^{\infty} E_x(0, \varphi, z = vt - v\tau, t) v dt$	$Z(\omega) = \frac{1}{Q} \int_{-\infty}^{\infty} \vec{E}_x(0, \varphi, z, \omega) e^{i(\omega/v)z} dz$

## TRANSFER TO A NUMERICAL FORMULATION

To calculate the continuous spectrum of the impedance in the frequency domain above cut-off, a computer code will be presented to evaluate the impedance values at any given frequency. For this

purpose Maxwell's equations have to be discretized and the resulting numerical problem has to be solved. Here finite differences, especially the FIT-method [4], are used for the discretization on a rectangular mesh, so that the program is based on the code URMEL [1]. Since only the cavity region and a piece of the (infinitely long) beam ports is entered as geometry an "open" boundary condition is necessary.

#### Maxwell's Equations for the problem

Because of the cylindrical symmetry the fields have an azimuthal periodicity  $\sim e^{im\varphi}$  and their time dependence is harmonic. At the present state only monopole fields ( $m=0$ ), which have just the components  $H_\varphi$ ,  $E_r$  and  $E_z$ , are treated.

With these assumptions the fields may be written as :

$$\begin{aligned}\vec{E}(r, \varphi, z, t) &= \text{Re}(\sqrt{Z_0} (\vec{E}'_r(r, z) + \vec{E}'_z(r, z)) e^{i(\omega t + \varphi)}), \\ \vec{H}(r, \varphi, z, t) &= \text{Re}(\sqrt{Y_0} \vec{H}'_\varphi(r, z) e^{i(\omega t + \varphi)}),\end{aligned}\quad (8)$$

with  $Z_0 = \sqrt{\mu_0/\epsilon_0}$  and  $Y_0 = 1/Z_0$  and complex  $\vec{E}'$  and  $\vec{H}'$ .

Consequently Maxwell's equations in the integral form that will be discretized read as :

$$\oint_{\partial A} \vec{H}' \cdot d\vec{s} = i \frac{\omega}{c} \cdot \int_A \epsilon_r \vec{E}' \cdot d\vec{A} + I', \quad (9)$$

$$\oint_{\partial A} \vec{E}' \cdot d\vec{s} = -i \frac{\omega}{c} \cdot \int_A \mu_r \vec{H}' \cdot d\vec{A}, \quad (10)$$

where  $\partial A$  stands for the boundary of area A.

Since the structure is excited by a beam current  $I'$  with frequency  $\omega$  the fields are composed of source fields and source-free fields, i.e. an inhomogeneous part that is caused by the current and the homogeneous part. So far only relativistic particles are considered, i.e.  $v = c$ .

The current  $I'$  is given by :

$$I'_z(\omega, r, z) = \begin{cases} c Q \sqrt{Z_0} e^{-ik(z-z_0)} & \text{for } r = 0 \\ 0 & \text{otherwise} \end{cases}, \quad (11)$$

with the wavenumber  $k = \omega/c$ .

This current induces the inhomogeneous magnetic field

$$H^*_\varphi(\omega, r, z) = \frac{c Q \sqrt{Z_0}}{2\pi r} e^{-ik(z-z_0)}, \quad (12)$$

with the phase  $kz_0$  and where  $H = H^*_\varphi + H^0_\varphi$ .

#### Treatment of the open boundary conditions

The cavity is excited by the current of angular frequency  $\omega$ . If  $\omega/2\pi$  lies above the lowest cut-off frequency for travelling modes in the beam port the induced fields won't stay in the cavity but will travel into the tubes to both sides, i.e. they are proportional to  $e^{ik'(z-z_0)}$  to the left of the cavity and  $e^{-ik'(z-z_0)}$  to the right of the cavity.  $k'$  is the propagation constant of the waves in the beam tubes. In the frequency range just above cut-off where only one type of waves can propagate  $k'$  is explicitly known. For higher frequencies  $k' = k$  is taken as approximation.

The geometric structure used for the numerical calculation has to have a finite length, i.e. the beam ports have to be cut at some convenient distance from the cavity. At these boundaries the reality of an open beam port has to be simulated. The difference equation for the radial electric field component at the left (right) boundary would need a value of  $H_\varphi$  and  $H^*_\varphi$  for an imaginary mesh point laying outside the grid (compare the next section). These values are calculated, as indicated below.

From (12) it follows for  $H^*_\varphi$

$$H^*_\varphi(\omega, r, z - \Delta) = H^*_\varphi(\omega, r, z) e^{ik\Delta} \doteq (1 + ik\Delta) H^*_\varphi(\omega, r, z) \quad (13)$$

For the homogeneous part of the azimuthal magnetic field the following relation is valid:

$$H^0_\varphi(\omega, r, z - \Delta) \doteq (1 - ik'\Delta) H^0_\varphi(\omega, r, z) \text{ at the left} \quad (14)$$

$$H^0_\varphi(\omega, r, z + \Delta) \doteq (1 - ik'\Delta) H^0_\varphi(\omega, r, z) \text{ at the right} \quad (15)$$

These first order equations are used to set up the difference equations for grid points at the right and left boundary, and they allow the simulation of an open boundary.

#### Discretization

Maxwell's equations are discretized with a special method of finite differences, the FIT-method [4], for a rectangular grid. This method uses two dual orthogonal grids and allocates the unknowns on these grids in a special manner so that the connection between area and line integrals is fulfilled in a natural way. Since the method presented here is based on URMEL [1], this paper will only point out the differences from URMEL and refers to the documentation about URMEL for more details.

The difference equation for  $H^0_\varphi$  is deduced from (10) while the discretization of equation (9) gives the difference equations for  $E_r$  and  $E_z$ . The setup of the equations for  $E_r$  at the boundary uses the open boundary condition (compare (13), (14) and (15)). The difference equations for  $E_z$  use for  $r > 0$  the relation  $r H^*_\varphi(\omega, r, z) = (r - \Delta) H^*_\varphi(\omega, r - \Delta, z)$  which holds because of (12); on the axis ( $r = 0$ ) the current does not vanish but can be expressed in terms of  $H^*_\varphi$ .

Thus the discretization of Maxwell's equations leads to difference equations for each unknown field component. Taking the one for  $H^0_\varphi$  and replacing the other components by their difference equations yields finally a system of linear equations with the homogeneous azimuthal field components as unknowns and the inhomogeneous ones as right hand sides.

### THE NUMERICAL PROBLEM

#### The matrix of the linear system

Discretization of Maxwell's equations leads to a complex linear system of equations

$$M \vec{h} = (A + ik'D - k^2 I) \vec{h} = k^2 \vec{h} \quad (16)$$

The matrix of this system has the dimension of  $N \times N$  where  $N$  stands for the number of mesh points. The solution vector  $\vec{h}$  is the vector  $(H^0_{\varphi 1}, \dots, H^0_{\varphi N})$  with the homogeneous azimuthal magnetic field components. The matrices A, D and I are purely real; the last one stands for the unit matrix. The matrix A equals the matrix of URMEL [1] for the monopole case. D is a diagonal matrix expressing the open boundary condition. The right hand side  $\vec{h}^*$  is the vector  $(H^*_{\varphi 1}, \dots, H^*_{\varphi N})$  with the inhomogeneous azimuthal magnetic field components.

The matrix A has a band structure with only four off-diagonals; obviously M has the same structure. Furthermore A can be made symmetric as is explained in [1] but M is not hermitian and not even positive definite which is needed by nearly all iterative numerical methods, i.e. a system with complex indefinite system matrix has to be solved.

#### Solution method for the linear system

The program consists mainly of three parts: The setup of the matrix, the solution of the linear system and the evaluation of the fields and the impedance. As a first step the first and last parts have received the greatest effort. The present version is limited to a few hundred mesh points because it uses a standard direct method which gives acceptable results for the solution of the linear system. The final program version will use a fast solver for the system of equations, which will allow as many points  $N$  as URMEL (e.g. 5000 on IBM, 2Mbyte) and, as it is intended, will also allow for dipole modes.

#### Evaluation of the impedance

The solution of the linear system gave  $H^0_\varphi$ . The electric field components are calculated from  $H^0_\varphi$  and  $H^*_\varphi$  by the difference equations. The impedance is given by (7). In [5] it is shown that the integral can be taken at any radius  $r$  for a cylindrically symmetric structure.

Thus integrating at the tube radius gives the same result as the integral on the axis. The advantage of integrating at the tube radius is that the integrand vanishes in the tube region. Consequently one needs only to perform the integral over the gap of the cavity.

**EXAMPLES**

Pillbox cavity

Recently H.Henke published a paper [6] in which he numerically calculates impedances for single pillboxes with side tubes by Fourier series. This was taken for comparisons showing a good agreement. As example a pillbox with 198 mm gap, 278 mm radius and 50 mm tube radius was chosen. The differences are caused by the coarseness of the mesh, which gets a growing influence with growing frequency. Thus it may be concluded that the method presented here works well and only the solution method for the linear system has to be improved to allow finer meshes.

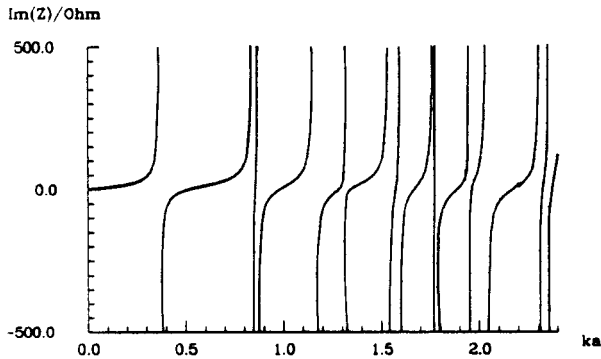


Figure 1 : Reactive impedance below cut-off computed by URMEL-I

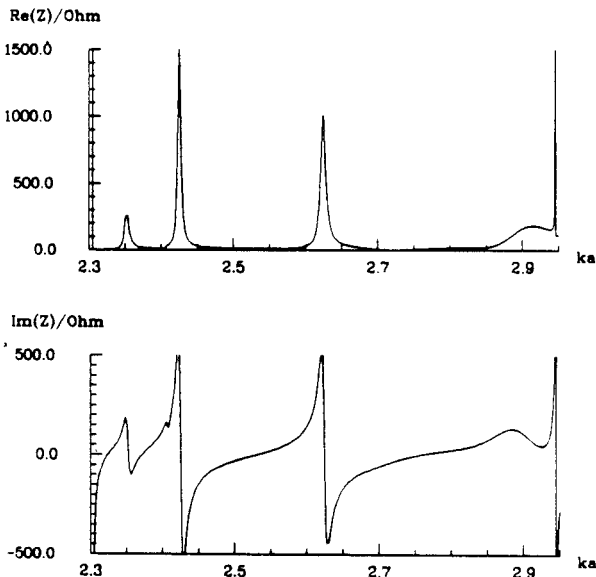


Figure 2 : Impedance above cut-off computed by URMEL-I

PETRA cavity

In general the cavities used in accelerators have a shape differing more or less from a pillbox. The results of the impedance calculations for the PETRA cavity are presented in the following figures. The lowest TM<sub>0</sub>-mode of this cavity equals 514 MHz, the cut-off frequency is 1912 MHz. The comparison with results obtained by TBCI [5] combined with Fast Fourier Transformation (FFT) shows that TBCI allows only a quite rough approximation of the impedance.

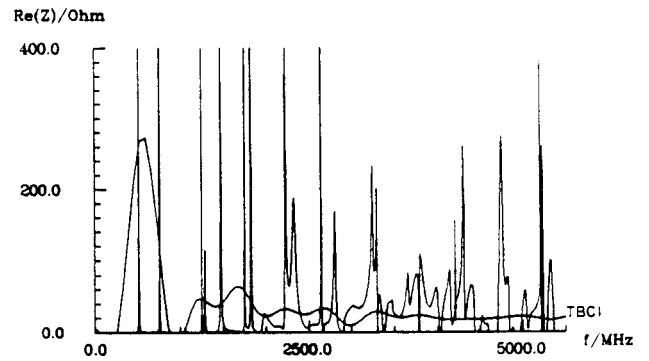


Figure 3 : Impedance of PETRA cavity computed by URMEL-I (divided by 2π) and computed by TBCI/FFT

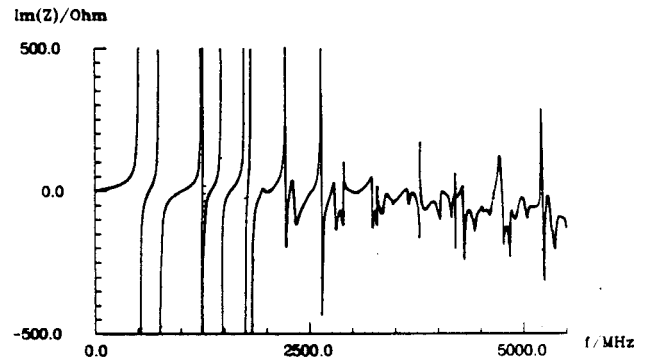


Figure 4 : Reactive impedance computed by URMEL-I (divided by 2π)

**SUMMARY**

The extension of the resonator code URMEL to URMEL-I presents a new and efficient tool to calculate the impedance of obstacles of arbitrary but cylindrically symmetric shape with side tubes. The preliminary version of this code gives good results, indicating the validity of the method.

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**REFERENCES**

- [1] T.Weiland, On the computation of resonant modes in cylindrically symmetric cavities, *NIM* 216 (1983), 329-348
- [2] T.Weiland, B.Zotter, Wake potentials of a relativistic current in a cavity, *Particle Accelerators* 11 (1981), 143-151
- [3] P.B.Wilson, High energy electron linacs: Applications to storage ring rf systems and linear colliders, SLAC-PUB-2884, February 1982
- [4] T.Weiland, Eine Methode zur Lösung der Maxwell'schen Gleichungen für sechskomponentige Felder auf diskreter Basis, *AEÜ Archiv für Elektronik und Übertragungstechnik* 31 (1977), 116-120
- [5] T.Weiland, Transverse beam cavity interaction, Part I: Short range forces, *NIM* 212 (1983), 13-21
- [6] H.Henke, Point charge passing a resonator with beam tubes, CERN-LEP-RF/85-41, November 1985