LONGITUDINAL COUPLING IMPEDANCE AND ITS HIGH FREQUENCY BEHAVIOR*

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I. Introduction

The high frequency behavior of the coupling impedance for an obstacle in a beam pipe is of considerable importance because of recent interest in the acceleration and transport of short beam bunches. Lawson[1] used a diffraction model to show that the real part of the longitudinal impedance falls off as $(\text{frequency})^{-1/2}$, a result confirmed by Dôme[2] by an approximate calculation for both the real and imaginary parts. Heifets and Kheifets[3] use an iteration method to confirm Dôme's result for the real part. Gluckstern and Zotter[4,5] derive an integral equation for the axial electric field at the beam pipe radius, whose solution is needed to obtain the impedance. In this paper we explore approximations to the kernel of this integral equation which permit us to predict the behavior of the impedance for small obstacles as well as for obstacles of arbitrary size at high frequency.

II. Analysis

The starting point for the analysis is the integral equation obtained for the electric field in the obstacle at the pipe radius.[5] Specifically, we have

$$\int_0^g dz'\, F(z') \left[K_p\left(|z'-z|\right) + K_c(z',z)\right] = j\, e^{-jkz} \tag{2.1}$$

and

$$\frac{Z(k)}{Z_o} = \frac{1}{ka^2}\int_0^g dz\, F(z)\, e^{jkz} \quad . \tag{2.2}$$

Here $kc/2\pi$ is the frequency, a is the pipe radius, $Z_o = 120\pi$ ohms is the impedance of free space, and the azimuthally symmetric cavity, of general shape in the r,z plane, extends axially from $z = 0$, to $z = g$ at the pipe radius $r = a$. Apart from a constant, $F(z)$ is the axial electric field for $r = a$ and $0 < z < g$. The component of the kernel from the pipe field is

$$K_p(|u|) = \frac{2\pi j}{a}\sum_{s=1}^{\infty}\frac{e^{-jb_s|u|/a}}{b_s} \quad , \tag{2.3}$$

where

$$b_s = \sqrt{k^2a^2 - j_s^2} \quad , \quad \beta_s = \sqrt{j_s^2 - k^2a^2}\,. \tag{2.4}$$

Here j_s is the sth zero of the Bessel function $J_o(x)$ and b_s is to be replaced by $-j\beta_s$ when $j_s > ka$. The component of the kernel from the "cavity fields" is

$$K_c(z,z') = 4\pi^2 \sum_{\ell}\frac{h_\ell(z)\; h_\ell(z')}{k^2 - k_\ell^2} \quad , \tag{2.5}$$

where the orthonormal (azimuthally symmetric) modes of the cavity (with an imaginary metal wall at $r = a$) are defined by

$$\nabla \times \vec{e}_\ell = k_\ell \vec{h}_\ell \quad , \quad \nabla \times \vec{h}_\ell = k_\ell\, \vec{e}_\ell \,, \tag{2.6}$$

and where

$$h_\ell(z) \equiv \left[h_\ell(a,z)\right]_\varphi$$

is the azimuthal component of the normalized magnetic field at $r = a$.

III. Approximation for a Small Obstacle

Gluckstern and Neri[6] have shown that the cavity kernel in Eq. (2.5) can be approximated for a small obstacle by

$$K_c(z',z) \cong \frac{2\pi}{k^2 a\Delta} \quad \text{or} \quad K_c(z',z) \cong \frac{2\pi}{kg}\cot k(b-a) \quad , \tag{3.1}$$

where Δ is the cross sectional area of the cavity of arbitrary shape, and where the second form is an even more accurate for a narrow pillbox cavity of outer radius b. They also show that the pipe kernel can be approximated by

$$K_p(|u|) \cong \frac{\pi j}{a}\left[1 + K_p^o + \frac{2j}{\pi}\ell n \frac{2}{k|u|C}\right] \quad , \tag{3.2}$$

where $\ell nC = .5772$ is Euler's constant and where

$$K_p^o = \lim_{|u|\to 0}\left[2\sum_{s=1}^{\infty}\frac{e^{-jb_s|u|/a}}{b_s} - H_o^{(2)}(k|u|)\right] \quad . \tag{3.3}$$

depends only on ka. They then obtain the integral equation

$$\int_0^1 dx'\, f(x')\left[K - \frac{2j}{\pi}\ell n|x'-x|\right] \cong 1 \tag{3.4}$$

and an expression for the impedance:

$$\frac{Z(k)}{Z_o} \cong \frac{1}{\pi ka}\int_0^1 dx\, f(x) \quad , \tag{3.5}$$

where the constant K is

$$K \equiv K_p^o - \frac{2j}{k^2\Delta} + 1 + \frac{2j}{\pi} \ln \frac{2}{kgC} \qquad (3.6)$$

and where

$$F(z') = \frac{a}{\pi g} f(x') \quad . \qquad (3.7)$$

Finally, they show that the solution of Eq. (3.4) is

$$f(x) = - \frac{A\, x^{-1/2} (1-x)^{-1/2}}{2\pi \ln 2} \quad , \qquad (3.8)$$

with

$$A^{-1} = \pi ka\, [K + 2j(\ln 4)/\pi] \quad , \qquad (3.9)$$

leading to the admittance

$$Z_o Y(k) \cong 2\pi ka \left[- \frac{j}{k^2\Delta} + \frac{1}{2} + \frac{K_p^o}{2} + \frac{j}{\pi} \ln \frac{8}{kgC} \right] \quad . \qquad (3.10)$$

If one uses an approximate form for K_p^o valid for small g by setting u = g in Eq. (3.3), one obtains

$$Z_o Y(k) \cong 2\pi ka \left[- \frac{j}{k^2\Delta} + \sum_{s=1}^{\infty} \frac{e^{-jb_s g/a}}{b_s} + \frac{j}{\pi} \ln 4 \right] \quad . (3.11)$$

We have thus obtained a separation of the admittance into a smooth imaginary term depending on the cavity parameters and a complex term depending primarily on ka (and only logarithmically on g).

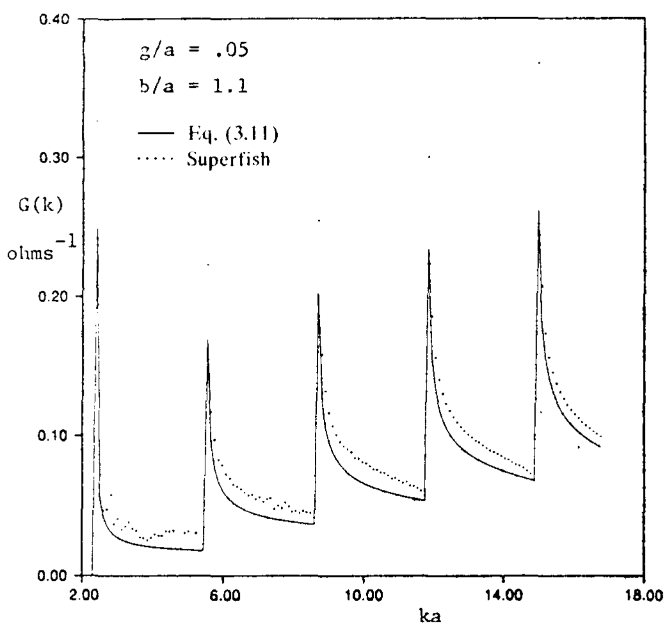

Figure 1

A comparison of Eq. (3.11), with the replacement $k^2\Delta = kg \tan k(b-a)$, to the results of a numerical program[7] for b/a = 1.1, g/a = .05 is shown is Figs. 1 and 2. The solid curves are the real and imaginary parts of Y(k) = G(k) + j B(k) obtained from Eq. (3.11), which, of course become infinite at ka = j_s. The dots correspond to the results of the numerical program.[7]

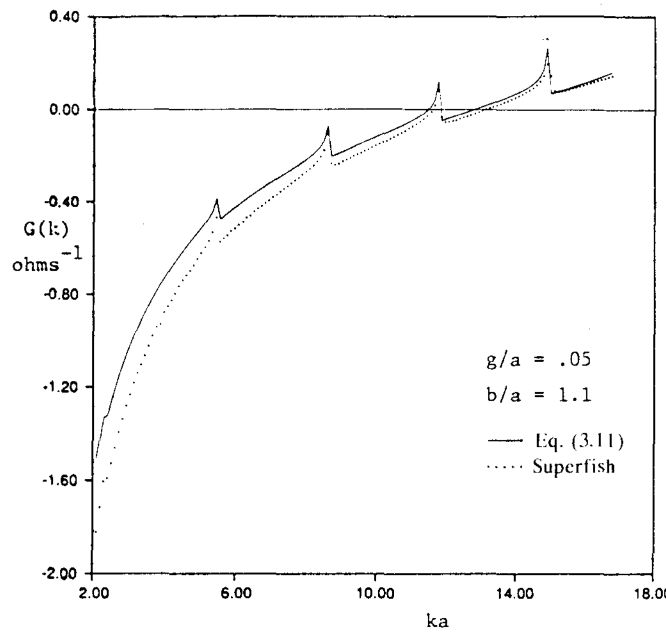

Figure 2

The broad resonance character[7] of the impedance of a small obstacle is apparent from the figure, particularly as ka is increased.

IV. Approximate Form for High Frequency

We can also solve the integral equation in Eq. (2.1) for large k for a specific cavity geometry. In this case one obtains expressions for both the pipe and cavity kernels by approximating the sums over s and ℓ by integrals. The result (for a pill-box cavity) is[5]

$$K_p(|z'-z|) \cong \frac{j\pi}{a} H_o^{(2)}(k|z'-z|) \quad ,$$

$$K_c(z',z) \cong \frac{j\pi}{a} H_o^{(2)}(k|z'-z|) \quad , \qquad (4.1)$$

leading to the integral equation

$$\int_o^g dz'\, G(z')\, e^{-jk(z'-z)}\, H_o^{(2)}(k|z'-z|) \cong 1 \quad , \qquad (4.2)$$

with the impedance becoming

$$\frac{Z(k)}{Z_o} = \frac{1}{2\pi ka}\int_o^g G(z)\, dz \quad . \qquad (4.3)$$

Since only a smooth variation of G(z) will contribute to the impedance, we approximate Eq. (4.2) for high k by using the asymptotic form of the Hankel function and neglecting rapidly varyng terms. The result is the equation

$$\int_o^z \frac{dz'\; G(z')}{\sqrt{z - z'}} \cong \frac{(1 - j)\sqrt{\pi k}}{2} \quad ,$$

whose solution is

$$G(z') \cong \frac{(1 - j)\sqrt{\pi k}}{2\pi \sqrt{z'}} \quad , \qquad (4.5)$$

leading to the impedance

$$\frac{Z(k)}{Z_o} \cong \frac{1 - j}{2\pi \sqrt{ka}}\left(\frac{g}{\pi a}\right)^{1/2} \quad . \qquad (4.6)$$

The high frequency dependence in Eq (4.6) agrees with that of D[me[2], and with that of Lawson[1] and of Heifets and Kheifets[3] who only obtain the real part.

The same result has been obtained[5] for upright and isosceles right triangles where the pipe kernel can also be shown to lead to the same approximate form in Eq. (4.1) for large k. It therefore appears that Eq. (4.6) is a general result for cavities of width g at the pipe radius, but otherwise arbitrary shape.

V. Causality

As a final point, the definition of Z(k) in terms of the wake function for an ultrarelativistic point charge implies that Z(k) is analytic in the lower half complex k plane, with

$$Z(-k) = Z^*(k) \quad . \qquad (5.1)$$

Use of the Hilbert transform pair allows us to write the real part of Z(k) as an integral of the imaginary part on the real axis, and vice-versa:

$$\text{Im } Z(k) = \frac{2k}{\pi} P \int_o^\infty \frac{dk'\; \text{Re}(k')}{k'^2 - k^2} \quad , \qquad (5.2)$$

$$\text{Re } Z(k) = -\frac{2}{\pi} P \int_o^\infty \frac{k'\; dk'\; \text{Im}(k')}{k'^2 - k^2} \quad . \qquad (5.3)$$

It is straightforward to show that the real and imaginary parts of Eq. (4.6) satisfy Eqs. (5.2) and (5.3) for large k, where the essential contributions to the integrals occur for k' of order k.

VI. References

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