BEAM BREAKUP WITH COUPLING BETWEEN CAVITIES*

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I. Introduction

Beam breakup in a multi-cavity linac is a phenomenon in which excitation in one cavity causes deflection of the beam, which then may enhance the excitation in a subsequent cavity. In cumulative beam breakup the cavities are identical and uncoupled. This phenomenon has been analyzed[3], yielding simple formulas for the transient behavior for a beam with an initial displacement. If the cavities are coupled, one needs to use the eigenmodes of the multi-cavity structure. If the modes are well separated, one has regenerative beam breakup, with the existence of a starting current above which the oscillations are unstable[4]. We develop the theory for the intermediate case, where the coupling constant, the relative mode separation and Q^{-1} are all comparable. The transition from cumulative to regenerative beam breakup is explored and compared to the results of simulations which include the coupling.

II. Beam Breakup Without Coupling

The analysis of cumulative beam breakup for a tightly bunched coasting beam starts with a set of difference equations for the displacement and angle of the Mth beam bunch entering the Nth cavity in terms of cavity excitation. An additional equation is needed for the change in excitation of the Nth cavity as a result of the transit of the Mth bunch. These equations can be combined into the following single difference equation for the displacement $\xi(N,M)$:

$$\xi(N+1,M) - 2\xi(N,M) + \xi(N-1,M) = r \sum_{\ell=0}^{M=1} \xi(N,\ell)\; s_{M-\ell} \qquad (2.1)$$

where

$$s_k = \exp(-k\omega\tau/2Q)\,\sin k\omega\tau \quad , \qquad (2.2)$$

$$r = \frac{eIc\tau}{2W}\left(\frac{Z_\perp}{LQ}\right) L^2 \quad . \qquad (2.3)$$

Here $\omega/2\pi$ is the deflecting mode frequency, I is the beam current for particles of relativistic energy W, L is the distance between cavity centers, $1/\tau$ is the bunch frequency, Q is the cavity quality factor and $Z_\perp$ is the transverse shunt impedance of the cavity (including transit time effects). In this paper we assume no transverse focussing. Gluckstern, Cooper and Channell[3] have obtained the solution to Eq. (2.1) and, from that, the following approximate expression for the transient response to a single bunch initially displaced by ξ_o:

$$\frac{\xi(N,M)}{\xi_o} \cong \frac{\exp(-M\omega\tau/2Q)}{M\sqrt{6\pi}}\,\mathrm{Re}\left[\sqrt{E}\,\exp(iM\omega\tau + 3E/2)\right] , \qquad (2.4)$$

where

$$E = r^{1/3}\, M^{1/3}\, N^{2/3}\, (\sqrt{3} - i)/2 \quad . \qquad (2.5)$$

Figure 1 contains a plot of $\xi(N,M)/\xi_o$ vs M obtained from a simulation for the values $r = 2.88 \times 10^{-3}$, $Q = 1000$, $\omega\tau/2\pi = 1.70$, $N = 30$ showing the transient amplification of the displacement by a factor of about 200. Equation (2.4) is an excellent representation of the result.

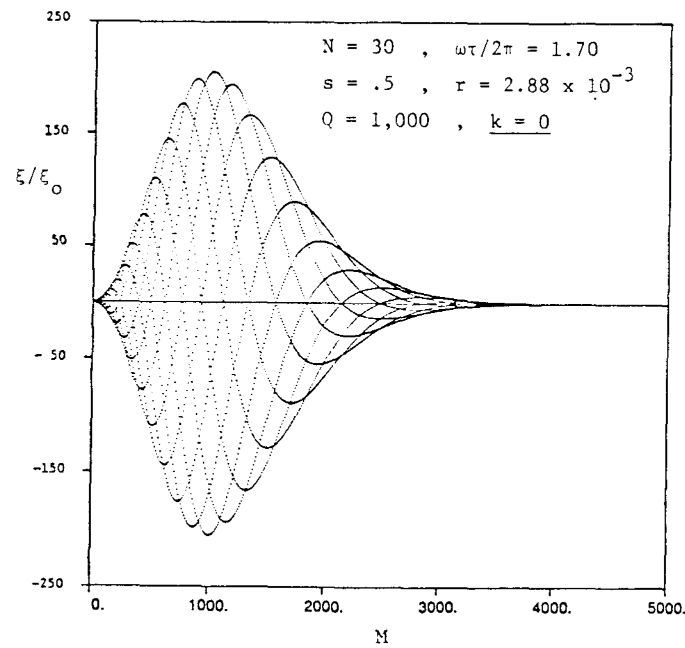

Figure 1

In a recent paper, Gluckstern, Neri and Cooper[5] explore the effect of smoothly varying parameters by starting with a pair of coupled differential equations as an approximation to Eq. (2.1) which is valid as long as $\omega\tau/2\pi$ is not an integer. Specifically they replace $\xi(N,M)$ by the complex function $z(N,M)$, with

$$\xi(N,M) = \mathrm{Re}\,[z(N,M)\,\exp(iM\omega\tau - \omega\tau/2Q)] \qquad (2.6)$$

and denote the cavity excitation (apart from a factor) by

$$\mathrm{Re}\,[v(N,M)]G3 \equiv \mathrm{Re}\,[u(N,M)\,\exp(iM\omega\tau - \omega\tau/2Q)] \quad . \qquad (2.7)$$

After averaging over rapid oscillations they obtain

$$\frac{\partial^2 z}{\partial N^2} \cong \frac{ru}{2i} \qquad (2.8)$$

$$\frac{\partial u}{\partial M} \cong z \quad . \qquad (2.9)$$

The asymptotic solution in Eq. (2.1) can be obtained[5] (apart from an overall factor) from these two equations for large M,N.

III. Coupling Between Cavities

The effect of coupling between cavities is introduced by modifying Eq. (2.9) to include the rate of change in excitation z(N,M) due to coupling from the adjacent cavities[6]. Specifically we start with the equation for the excitation due to coupling alone:

$$\frac{\partial^2 v_N}{\partial t^2} + \frac{1}{Q}\frac{\partial v_N}{\partial t} + \omega^2 v_N = -\frac{k\omega^2}{2}\left(v_{N+1} + v_{N-1}\right) , \quad (3.1)$$

where k is the coupling constant. It should be noted that the variables in Eq. (3.1) are the time t and cavity number N, rather than M and N as in Eq. (2.9). We can convert variables to M and N in Eq. (3.1) by using the relation

$$t = M\tau + NL/c = (M + Ns)\,\tau \quad , \quad (3.2)$$

where $s = L/c\tau$, the ratio of cavity spacing to bunch spacing. This implies that $v_{N\pm1}$ on the right side of Eq. (3.1) must be evaluated at bunch number $M\mp s$ to correspond to the time t in Eq. (3.2). If one changes variable from $v_N(t)$ to u(N,M) according to Eqs. (2.7) and (3.2), and assumes that u(N,M) is a slowly varying function of N and M, one finds

$$\frac{\partial u}{\partial M} \cong \frac{ik\omega\tau}{4}\left[u(N+1,M-s)\,e^{-i\Theta} + u(N-1,M+s)\,e^{i\Theta}\right] , \quad (3.3)$$

where $\Theta = s\omega\tau$.

The effect of coupling between adjacent cavities on beam breakup can therefore be included by adding the right sides of Eqs. (2.9) and (3.3) to obtain

$$\frac{\partial u}{\partial M} \cong z(N,M) + \frac{ik\omega\tau}{4}\left[u(N+1,M-s)e^{-i\Theta} + u(N-1,M+s)e^{i\Theta}\right] . \quad (3.4)$$

Equations (2.8) and (3.4) therefore represent our model for beam breakup with coupling between cavities.

IV. Solution for Small Coupling

For small coupling, the term proportional to k in Eq. (3.4) will be small, and the solution will be a small modification to that corresponding to Eq. (2.4). In this case s can be neglected in Eq. (3.4) and one can approximate $u(N\pm1,M)$ by the first term in a Taylor expansion around u(N,M) obtaining

$$\frac{\partial u}{\partial M} \cong z + i\Delta u + \varepsilon\,\frac{\partial u}{\partial N} \quad , \quad (4.1)$$

where

$$2\Delta = k\omega\tau\cos\Theta \quad , \quad 2\varepsilon = k\omega\tau\sin\Theta \quad . \quad (4.2)$$

The term in Δ corresponds simply to a frequency change which can be removed by absorbing the factor $\exp(iM\Delta)$ into u and z, which are now denoted by $\tilde{u}$ and $\tilde{z}$. If one changes variables from M and N to m=M and $n=N+\varepsilon M$, Eqs. (2.9) and (4.1) become

$$\frac{\partial\tilde{u}}{\partial m} = \tilde{z} \quad , \quad \frac{\partial^2\tilde{u}}{\partial n^2} = \frac{r}{2i}\,\tilde{u} \quad , \quad (4.3)$$

identical in form to Eqs. (2.8) and (2.9). The real part of the exponent, including that corresponding to Eqs. (2.6) and (2.7), is therefore

$$\exp = \frac{-M\omega\tau}{2Q} + \frac{3\sqrt{3}}{4}\,r^{1/3}M^{1/3}(N + \varepsilon M)^{2/3} \quad (4.4)$$

which, to first power in ε is

$$\exp \cong \frac{-M\omega\tau}{2Q} + \frac{3\sqrt{3}}{4}\,r^{1/3}M^{1/3}N^{2/3} + \varepsilon\,\frac{\sqrt{3}}{2}\,r^{1/3}M^{4/3}N^{-1/3} \quad . \quad (4.5)$$

We have obtained the exponent numerically from the envelope of the simulations and, after subtracting the first two terms on the right side of Eq. (4.5), have shown that what remains is linear with $M^{4/3}$ and proportional to k, confirming the form of Eq. (4.5). From this we conclude that beam breakup is enhanced for small positive ε and suppressed for small negative ε, which is confirmed by the simulations. In fact, for the parameters used in Fig. 1, we find major modifications (instability) with $k \geq .0045$ and $-k \geq .0015$. These are remarkably small coupling constants for instability, suggesting the need to include coupling in the deflecting mode in the design of structures like side-coupled electron linacs.

An interesting feature of Eq. (4.4) is that as $M \to \infty$, there will be a runaway oscillation if

$$\frac{81\sqrt{3}}{8}\,r\,\varepsilon^2 > \left(\frac{\omega\tau}{Q}\right)^3 \quad , \quad (4.6)$$

corresponding to the definition of a "starting current", confirmed by simulations only for positive ε.[7]

V. Solution for Finite N

It is possible to solve Eqs. (2.8) and (3.4) by writing the solution to Eq. (2.8) (in its finite difference form) as

$$z(N,M) = \frac{r}{2i}\left[u(N-1,M) + 2u(N-2,M) + 3u(N-3,M) + \cdots\right] \quad (5.1)$$

and trying solutions of the form exp(pM) to obtain the eigenmodes corresponding to p as solutions of an NxN determinantal equation. The solution corresponding to a starting current occurs for the smallest value of r for which the value of p with the largest real part is equal to $\omega\tau/2Q$. The form of the elements of the determinant suggest that the dimensionless parameter $y = rQ/\omega\tau$ should be a simple function of $x = 1/|k|Q$, more or less independent of other combinations of parameters. That this is the case is shown for various values of k in Figs. 2 and 3 for N=2 and N=15 respectively. These figures illustrate the difference between positive and negative k.

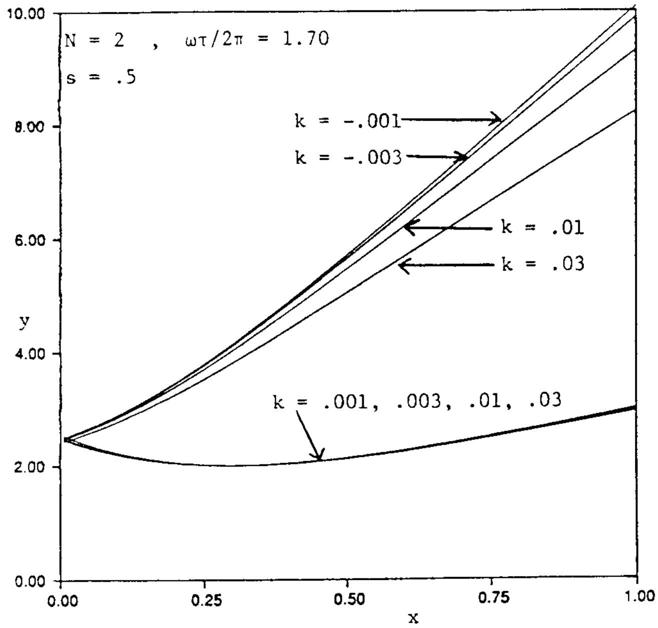

Figure 2

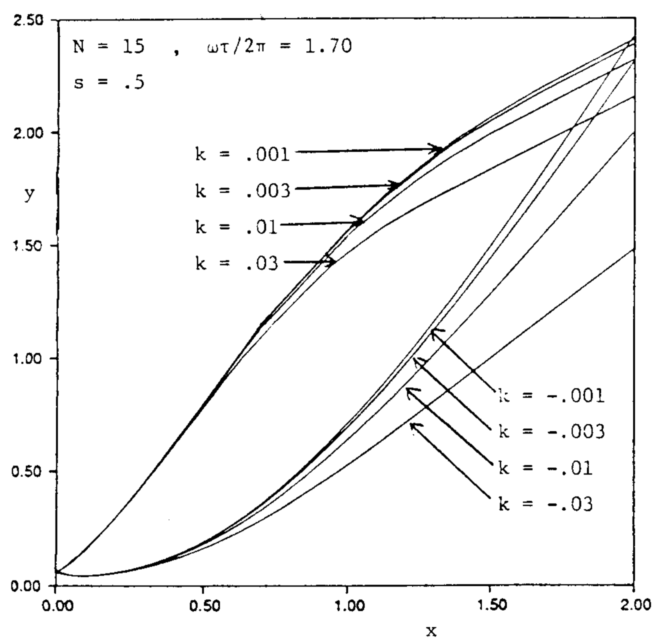

Figure 3

VI. Regenerative Beam Breakup

The concept of regenerative beam breakup involves the interaction of a beam with the transverse modes of a long cavity. The analysis of Wilson[4] leads to a starting current given by

$$y = \frac{\pi^3}{2N^2\, g(\alpha)} \qquad (6.1)$$

for a cavity consisting of N coupled cells, with $g(\alpha) = (\pi^3/2\alpha^2)(1 - \cos\alpha - \alpha \sin\alpha/2)$ having a maximum of 1.05 at $\alpha = 2.65$. In the analysis, Wilson assumes that all cavity modes are isolated from one another, that is the relative width of a mode, Q^{-1}, is smaller than the relative separation of modes, $|k|/N$.

In the language of the previous section, the parameter x takes on a useful physical meaning. For cavity modes which are well separated, $x = 1/|k|Q \ll 1/N$. For a band whose relative width, k, is smaller than Q^{-1} all modes in the band contribute at one time, corresponding to the cumulative beam breakup limit where $x = 1/|k|Q \gg 1$. Thus the intermediate case corresponds to the range $1/N < x < 1$, which is the region where we have derived the more or less universal curves shown in Figs. 2 and 3.

As a final point, we have solved[6] the determinantal equation outlined in Section V for small x, and reproduce Eq. (6.1) for the regenerative beam breakup limit. Thus we have developed a formalism which includes both the cumulative beam breakup and regenerative beam breakup limits and which gives more or less universal results in the intermediate region between these limits.

VII. References

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4. See, for example, P.B. Wilson, Proceedings of the Fermilab Summer School on Particle Accelerators, p. 450 (1981); R.H. Helm and G.A. Loew, Linear Accelerators, edited by P. Lapostolle and A. Septier, John Wiley and Sons, p. 173 (1970).

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6. Gluckstern, Neri and Cooper, "Beam Breakup with Coupling Between Cavities", submitted for publication.

7. For reasons related to N being finite, as explained in Reference 6.