USING TRAVELING WAVE STRUCTURES TO EXTRACT POWER FROM RELATIVISTIC KLYSTRONS*

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Abstract

The purpose of this note is to analyze the excitation of traveling wave (TW) output structures by an RF current. Such structures are being used in relativistic klystron experiments at Lawrence Livermore National Laboratory. First we will present a set of difference equations that describes the excitation of the cells of a TW structure. Next we will restrict our attention to structures that have identical cells, except possibly for the first and last cells. Under these circumstances one can obtain difference equations that have constant coefficients, and we will present the general solution of these equations. Lastly we will apply our results to the analysis of a TW output structure. We will show that, by appropriate choice of the quality factors (Qs) and eigenfrequencies of the first and last cells, it is possible to obtain a traveling wave solution for which there is no reflected wave and where the excitation grows linearly with cell number.

Difference Equation Formulation

Consider a TW structure consisting of N cells. Let the electric field in the n^{th} cell of the structure be given by

$$\vec{E}_n(\vec{r},t) = a_n(t)\vec{\mathcal{E}}_n(\vec{r})e^{-im\omega t}, \tag{1}$$

where $\vec{\mathcal{E}}_n$ denotes the eigenmode of the n^{th} cell with eigenfrequency ω_n, and where we have assumed that $\omega_n \approx m\omega$. It is possible to show that, in the steady state, the excitations a_n are governed by the following difference equations:

$$\left(\omega_n^2 - m^2\omega^2 - i\frac{m\omega\omega_n}{Q_n}\right)a_n - \left(K_n^{n-1}a_{n-1} + K_n^{n+1}a_{n+1}\right) = \frac{im\omega}{\epsilon_o}\int dV_n \vec{\mathcal{E}}_n^* \cdot \vec{J}_m, \tag{2}$$

where $n = 1,\ldots,N$, and where $y_0 = y_{N+1} = 0$. In the above equations, K_n^{n-1} and K_n^{n+1} describe the coupling of cell n to cell $n-1$ and cell $n+1$, respectively. The quantity Q_n denotes the quality factor the n^{th} cell. For a klystron output cavity, $\vec{J}_m$ denotes a component of the RF current associated with the bunched beam. It is important to note that the power extracted from the TW structure is proportional to $|a_N|^2$.

Below we will show that, by appropriate choice of Q_1, Q_N, ω_1 and ω_N, it is possible to obtain a solution of the difference equations that represents a traveling wave whose amplitude grows linearly with cell number.

To simplify our analysis, we will suppose that

$$K_n^{n-1} = K_n^{n+1} \stackrel{def}{=} K. \tag{3}$$

Then the difference equations have the following form:

$$\left(\omega_n^2 - m^2\omega^2 - i\frac{m\omega\omega_n}{Q_n}\right)y_n - K(y_{n-1} + y_{n+1}) = I_n, \tag{4}$$

where $n = 1,\ldots,N$ and where

$$y_0 = y_{N+1} = 0. \tag{5}$$

Now suppose that, except for the first and last cells, the quantities Q_n and ω_n are the same for all cells:

$$\begin{aligned} Q_n &= Q_\nu \quad (n = 2,\ldots,N-1), \\ \omega_n &= \omega_\nu \quad (n = 2,\ldots,N-1). \end{aligned} \tag{6}$$

We will allow for the possibility that Q_1 and ω_1 may be different; this would be the case if, for example, there were a coupling hole for power input. Also, we will allow for the possibility that Q_N and ω_N may be different; this would be the case if, for example, there were a coupling hole for power output. It follows that the difference equations can be written in the following form:

$$y_{n+1} - 2y_n\cos\alpha + y_{n-1} = f_n \quad (n = 1,\ldots,N), \tag{7}$$

where

$$\cos\alpha = \frac{1}{2K}\left(\omega_\nu^2 - m^2\omega^2 - i\frac{m\omega\omega_\nu}{Q_\nu}\right), \tag{8}$$

$$f_n = -I_n/K, \tag{9}$$

and with boundary conditions

$$\begin{aligned} y_0 &= \mu y_1, \\ y_{N+1} &= \eta y_N, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \mu &= \frac{1}{K}\left[\left(\omega_\nu^2 - i\frac{m\omega\omega_\nu}{Q_\nu}\right) - \left(\omega_1^2 - i\frac{m\omega\omega_1}{Q_1}\right)\right], \\ \eta &= \frac{1}{K}\left[\left(\omega_\nu^2 - i\frac{m\omega\omega_\nu}{Q_\nu}\right) - \left(\omega_N^2 - i\frac{m\omega\omega_1}{Q_N}\right)\right]. \end{aligned} \tag{11}$$

(We usually deal with structures that have $Q_\nu \gg 1$. In this case α has a small imaginary part that can be neglected

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if the total number of cells is not too large. In view of equation (8), the quantity ω_ν is the frequency for which the phase shift per cell equals $\pi/2$.)

The important point is that we have obtained difference equations with *constant coefficients* (at the expense of more complicated boundary conditions). These equations can be easily solved. The general solution of (7) is given by[1]

$$y_n = e^{i\alpha n}\left[\frac{-i}{2\sin\alpha}\sum_{r=1}^{n} f_r e^{-i\alpha r} + c_1\right] + e^{-i\alpha n}\left[\frac{i}{2\sin\alpha}\sum_{r=1}^{n} f_r e^{i\alpha r} + c_2\right], \tag{12}$$

where c_1 and c_2 are constants. In the above expression, the following is assumed:

$$\sum_{r=1}^{0} f_r e^{\pm i\alpha r} = 0. \tag{13}$$

Also, we assume that α is not equal to an integral multiple of π.

Application to a Traveling Wave Output Structure

Now we will apply these results to a TW output structure. Suppose the drive terms, f_n, have constant amplitude, but that the phase of f_n increases by α from cell to cell:

$$f_n = f e^{i\alpha n}. \tag{14}$$

It follows that

$$\sum_{r=1}^{n} f_r e^{-i\alpha r} = nf, \qquad \sum_{r=1}^{n} f_r e^{i\alpha r} = f e^{2i\alpha}\frac{1-e^{2i\alpha n}}{1-e^{2i\alpha}}. \tag{15}$$

It is easy to show that the solution of the difference equations is now given by

$$y_n = e^{i\alpha n}\left[\left(\frac{-i}{2\sin\alpha}\right)nf + \frac{fe^{i\alpha}}{4\sin^2\alpha} + c_1\right] + e^{-i\alpha n}\left[-\frac{fe^{i\alpha}}{4\sin^2\alpha} + c_2\right]. \tag{16}$$

Since we have assumed that the fields vary as $e^{-im\omega t}$, the first term above represents a wave traveling in the forward direction (from $n = 1$ to $n = N$), and the second term represents a wave traveling in the backward direction.

Now we must apply the boundary conditions given by (10) and (11). After some manipulation, the condition relating y_0 and y_1 leads to the following:

$$\mu = \frac{c_1 + c_2}{c_1 e^{i\alpha} + c_2 e^{-i\alpha}}. \tag{17}$$

The condition relating y_N and y_N+1 is more complicated. However, we shall assume that

$$\cos N\alpha = 1, \qquad \sin N\alpha = 0. \tag{18}$$

In this case we obtain

$$\eta = \frac{\frac{Nf}{2i\sin\alpha}e^{i\alpha} + c_1 e^{i\alpha} + c_2 e^{-i\alpha}}{\frac{Nf}{2i\sin\alpha} + c_1 + c_2}. \tag{19}$$

Equations (17) and (19) are simultaneous equations in c_1 and c_2. Their solution is given by

$$c_1 = \left(\frac{-Nf}{4\sin^2\alpha}\right)\frac{(1-\mu e^{-i\alpha})(\eta - e^{i\alpha})}{1-\mu\eta}, \qquad c_2 = \left(\frac{Nf}{4\sin^2\alpha}\right)\frac{(1-\mu e^{i\alpha})(\eta - e^{i\alpha})}{1-\mu\eta}. \tag{20}$$

Summarizing, the solution to our problem is given by equation (16), where c_1 and c_2 are given by (20). The solution depends on the quantities μ and η. Assuming that Q_ν and ω_ν are given, the quantities μ and η are determined by the choice of Q_1, Q_N, ω_1 and ω_N through (11).

Matching Condition

By appropriate choice of μ and η it is possible to obtain a solution which consists of just a forward traveling wave. (This is what we mean by a "matched" structure). Referring to (16), the backward traveling wave will vanish when

$$c_2 = \frac{fe^{i\alpha}}{4\sin^2\alpha}. \tag{21}$$

Substituting equation (20) for c_2, we obtain the following matching condition:

$$N(1-\mu e^{i\alpha})(\eta - e^{i\alpha}) = (1-\mu\eta)e^{i\alpha}, \tag{22}$$

or

$$\mu e^{i\alpha} - \mu\eta\left(\frac{N-1}{N}\right) + \eta e^{-i\alpha} = \frac{N+1}{N}. \tag{23}$$

It follows that the solution of the difference equations is given by

$$y_n = \frac{fe^{i\alpha n}}{2i\sin\alpha}\left[n + \frac{\mu e^{i\alpha}}{1-\mu e^{i\alpha}}\right]. \tag{24}$$

The above equation shows that the excitation grows linearly with cell number in a matched TW structure.

Now consider the amplitude of the excitation in two extreme cases, $\mu = 0$ and $\mu \to \infty$:

$\mu = 0$

This corresponds to the case where the first cell has the same Q and ω as cells 2 through N. In this case, the matching condition reduces to

$$\eta = \left(\frac{N+1}{N}\right)e^{i\alpha}. \tag{25}$$

When $\mu = 0$, the constant term in square brackets in equation (24) (that is, the term that is independent of n) equals zero. In other words, the quantity in square brackets is equal to n.

$|\mu| \rightarrow \infty$

This corresponds to the case where $Q_1 \ll 1$. In this case, the matching condition reduces to

$$\eta = \left(\frac{N}{N-1}\right) e^{i\alpha}. \tag{26}$$

When $|\mu| \rightarrow \infty$, the constant term in (24) approaches -1, and the resulting quantity in square brackets approaches $n-1$. Therefore the excitation of the n^{th} cell is reduced from the $\mu = 0$ value by a factor $(n-1)/n$.

Obviously the true value of μ will lie between the extremes discussed above. Since the power produced by the structure is proportional to $|y_N|^2$, lowering the Q of the first cell will reduce the power output by a factor of at most $[(N-1)/N]^2$ from the maximum obtainable ($\mu = 0$) value.

Summary

The purpose of this note has been to analyze the excitation of TW output structures for use in relativistic klystrons. First we presented a set of difference equations that describes the excitation of the cells of a TW structure. Next we restricted our attention to structures that have identical cells, except possibly for the first and last cells. Under these circumstances, we were able to obtain difference equations that have constant coefficients, and we presented the general solution of these equations. Lastly we applied our results to the analysis of a TW output structure. We showed that, by appropriate choice of the quality factors and eigenfrequencies of the first and last cells, it is possible to obtain a traveling wave solution for which there is no reflected wave and where the excitation grows linearly with cell number.

References

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