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# ASYMPTOTIC ANALYSIS OF THE LONGITUDINAL INSTABILITY OF A HEAVY ION INDUCTION LINAC*

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## Abstract

An Induction Linac accelerating high ion currents at sub-relativistic energies is predicted to exhibit unstable growth of current fluctuations at low frequencies. The instability is driven by the interaction between the beam and complex impedance of the induction modules. In general, the detailed form of the growing disturbance depends on the initial perturbation and ratio of pulse length to accelerator length, as well as the specific form of the impedance. An asymptotic analysis of the several regimes of interest is presented.

## Linac Model

We treat a cluster of beams drifting at velocity $v$, with line charge density $\lambda$ and current $I = \lambda v$. It is assumed here that all the beamlets $(N \sim 16)$ effectively act in concert so that $\lambda$ and $I$ are the total values and $v$ is the common velocity. The continuity equation, written in laboratory frame quantities $(z,t)$ is:

$$\frac{\partial \lambda}{\partial t} + \frac{\partial I}{\partial z} = 0 \ . \tag{1}$$

A smoothed longitudinal field $E$, induced by interaction of $I$ with the induction modules, acts on $v$:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = \frac{qe}{m} E \ . \tag{2}$$

In general $E$ is related to $I$ through an impedance

$$E(\omega) = -Z(\omega)\, I(\omega) \ . \tag{3}$$

However in the present study the low frequency interaction is modeled as that of a resistance $R$ and capacity $C$ in parallel. We may use the circuit representation:

$$\frac{E}{RC} + \frac{\partial E}{\partial t} = -\frac{I}{C} \tag{4}$$

Most previous related work(1) has neglected the capacity, but included a direct space-charge force proportional to $\partial\lambda/\partial z$ The present model appears to be more representative at low frequencies for the Heavy Ion Fusion application. In general the capacity reduces growth rates compared with the case of pure resistance by lowering the impedance as frequency increases.

A perturbation analysis is carried out for small variations from constant equilibrium values. For:

$$v = v_o + \delta v \ ,$$
$$\lambda = \lambda_o + \delta\lambda \ ,$$
$$I = I_o + \delta I \ ,$$
$$E = \delta E \ ,$$

the perturbed equations are:

$$\delta I = \lambda_o \delta v + v_o \delta\lambda \ , \tag{5}$$

$$\frac{\partial \delta\lambda}{\partial t} + \frac{\partial \delta I}{\partial z} = 0 \ , \tag{6}$$

$$\frac{\partial \delta v}{\partial t} + v\frac{\partial \delta v}{\partial z} = \frac{qe}{m}\delta E \ , \tag{7}$$

$$\frac{\delta E}{RC} + \frac{\partial \delta E}{\partial t} = -\frac{\delta I}{C} \ . \tag{8}$$

The values of $R$ and $C$ are related to beam parameters by considerations of system efficiency. For a good match of source to beam load, $R$ must not be too different from the matched value $R_o = G/I_o$, where G is the average accelerating gradient. For the typical parameters $G = 10^6$ volts/m and $I_o = 1000$ amp, we have $R_o = 1000\ \Omega$/m. In this case R could be reduced to 300 $\Omega$/m without serious loss of efficiency. The characteristic time $RC \equiv \alpha^{-1}$ should be a small fraction of the pulse length to avoid excessive energy flow in charging the accelerating gaps. For the typical value $C = 3 \times 10^{-10}$ F-m, we have $RC = 90$ ns, which is short compared with a typical 500 ns pulse length.

In this simple model, time scales with $RC = \alpha^{-1}$, where the "retarded time" variable $\tau = t - z/v_o$ is used. A second scale quantity

$$K = \sqrt{\frac{qe\lambda_o}{m v_o^2 C}}$$

appears in the theory and scales the variable $z$. That is, $\alpha\tau$ and $Kz$ appear in a dimensionless formulation of Eqs. (5-8).

## Perturbation Analysis

If we neglect the self-force from space charge, proportional to $\partial\lambda/\partial z$, the coupled equations for perturbed field and current are conveniently written using $z$ and the retarded time $\tau = t - z/v_o$ as independent variables. We have

$$\frac{\partial^2 \delta I}{\partial z^2} = K^2 C \frac{\partial \delta E}{\partial \tau} \ , \tag{9}$$

$$\left(\alpha + \frac{\partial}{\partial \tau}\right)\delta E = -\frac{\delta I}{C} \ . \tag{10}$$

Initial conditions on $\delta I$ are specified at $z = 0$ for $\tau \geq 0$; in this model no disturbance is able to propagate backwards into the zones, $\tau < 0$ or $z < 0$. If the initial perturbation is a time-dependent velocity error generated at $z = 0$, the initial conditions are:

$$\delta I(o,\tau) = 0 \ ,$$
$$\delta E(z,o) = 0 \ ,$$
$$\frac{\partial}{\partial z}\delta I(o,\tau) \equiv f(\tau) = \frac{\lambda_o}{v_o}\frac{\partial \delta v(o,\tau)}{\partial \tau} . \tag{11}$$

The solution is now found with the aid of a Laplace transformation in $z$:

$$\left(\tilde{\delta I}, \tilde{\delta E}\right) = \int_o^\infty dz \exp(i\Omega z)\,(\delta I, \delta E) \ . \tag{12}$$

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Equations (9) and (10) yield

$$C\widetilde{\delta E} = \int_o^{\tau} d\tau' \frac{f(\tau')}{\Omega^2-K^2} \exp\left[\frac{\alpha\Omega^2}{\Omega^2-K^2}(\tau'-\tau)\right] , \qquad (13)$$

$$\delta\tilde{I} = -\left(\alpha + \frac{\partial}{\partial\tau}\right) C\widetilde{\delta E} , \qquad (14)$$

with inversion formula

$$\left(\delta I, \delta E\right) = \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} \exp(-i\Omega z)\, (\delta\tilde{I}, \delta\tilde{E}) . \qquad (15)$$

The inversion contour runs above any singularities in the complex $\Omega$ plane.

It is instructive to examine the case of an impulsive perturbation $f(\tau)$ resulting from a velocity step of amplitude $Av_oH(\tau-\tau_o)$; from Eq. (11):

$$f(\tau) = A(\tau_o)\lambda_o\delta(\tau-\tau_o) .$$

Then Eq. (13) gives

$$C\widetilde{\delta E} = \frac{A\lambda_o H(\tau-\tau_o)}{\Omega^2-K^2} \exp\left[-\frac{\alpha\Omega^2}{\Omega^2-K^2}(\tau-\tau_o)\right] . \qquad (16)$$

The inversion may be written

$$C\delta E = A\lambda_o H(\tau-\tau_o) \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} \frac{\exp(g)}{\Omega^2-K^2} , \qquad (17)$$

where we have defined

$$g(\Omega,\tau,z) = -i\Omega z - \frac{\alpha\Omega^2(\tau-\tau_o)}{\Omega^2-K^2} . \qquad (18)$$

The function [Eq. (17)] has been evaluated analytically for positive $(\tau,z)$, and may be regarded as a Green's function for general $f(\tau_o)$. However, its complicated form does not give a qualitative description of the pattern of growth with z and $\tau$. The saddle point analysis presented here, while inexact, does provide this picture in several regimes of $(z,\tau)$. We set $\tau_o = o$ in the following.

Note that there are poles in Eq. (17) on the real axis at $\Omega = \pm K$. These points are intrinsic singularities since the poles also appear in g. These singularities are associated with "mountain ranges" containing saddle points as displayed in Figs. 1 and 2. An asymptotic evaluation of $\delta E$ may be performed by the standard path of steepest descent method applied around the these points.

## Saddle Point Analysis

The stationary (saddle) points of $g(\Omega)$ are the four solutions $(\Omega_S)$ of the quartic equation

$$0 = \left(\frac{\partial g}{\partial\Omega}\right)_S = -iz + \frac{2\alpha\tau K^2\Omega_s}{\left(\Omega_s^2-K^2\right)^2} .$$

In general, two solutions lie in the lower half-plane and make little contribution to $\delta I$. The pair in the upper half-plane are found to be pure imaginary for $\Delta \equiv \alpha\tau/Kz > 8/(3\sqrt{3})$ and complex for $\Delta < 8/(3\sqrt{3})$ (see Figs. 1 and 2).

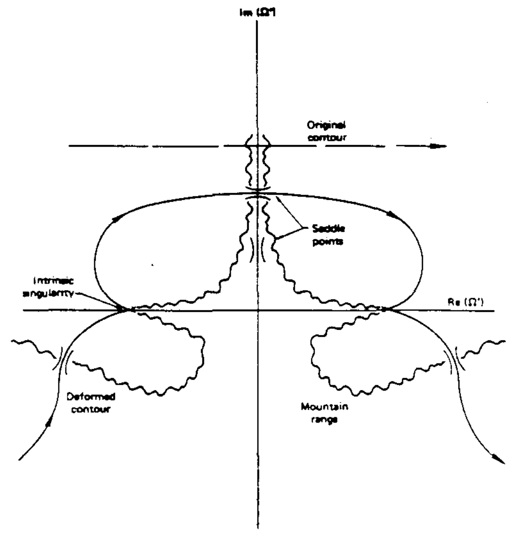


Fig 1. Topography of the $\Omega'$ plane ($\Omega' = \Omega/K$) for $\alpha\tau/Kz > 8/(3\sqrt{3})$.

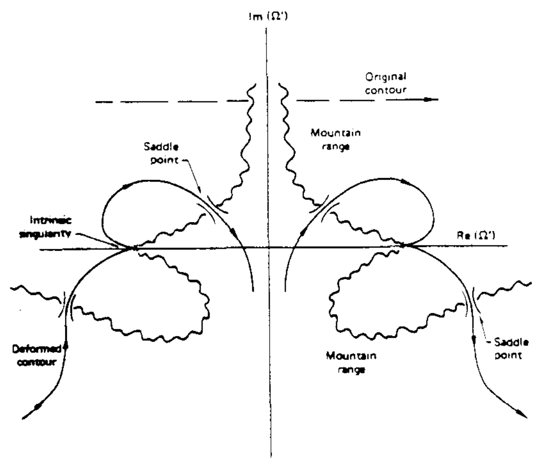


Fig 2. Topography of the $\Omega'$ plane for $\alpha\tau/Kz < 8/(3\sqrt{3})$.

Denoting $\Omega_S = K(\psi+i\mu)$, we have the parameterization in terms of $\mu$:

<u>Class 1</u>

$$\Delta > 8/(3\sqrt{3}), \quad \mu > 1/\sqrt{3} ,$$

$$\Delta = \left(\mu^2+1\right)^2/2\mu , \qquad (20)$$

$$g_s = \mu Kz - \frac{\mu^2}{\mu^2+1}\alpha\tau .$$

Inversion contour is horizontal through the upper saddle.

Class 2

$$\Delta < 8/(3\sqrt{3}), \quad \mu < 1/\sqrt{3} \ ,$$

$$\Delta = 4\mu^2 \sqrt{\mu^2 + 1} \ ,$$

$$\psi = \pm\left[\mu^2 + 1 - 2\mu\sqrt{\mu^2+1}\right]^{1/2} , \qquad (21)$$

$$\mathrm{Re}(g_s) = \mu Kz - \alpha\tau + \frac{\Delta\alpha\tau/2\mu}{(\Delta/2\mu)^2 + 4\mu^2\psi^2} \ .$$

Inversion contour is oblique through each saddle point.

The special case of $\Delta = 8/(3\sqrt{3})$ has the upper half plane saddles coalesce at $\mu = 1/\sqrt{3}$.

Case of $\Delta << 1$

Here the saddles lie close to the singularities $\Omega = \pm K$. Physically, z is large enough that resonant growth can dominate at small $\tau$. The quartic for $\Omega_s$ is readily solved by iteration from these values to obtain

$$\Omega_s \approx \pm K\left(1 \pm \frac{1+i}{2}\sqrt{\Delta}\right)$$

$$g_s \approx \pm iKz\left(1-\sqrt{\Delta}\right) + \sqrt{Kz\alpha\tau} - \frac{3}{4}\alpha\tau \ . \qquad (22)$$

Case of $\Delta >> 1$

From Eq. (20) we have

$$\mu \approx (2\Delta)^{1/3} ,$$

$$g_s \approx \frac{3}{2}(2\Delta)^{1/3} Kz - \alpha\tau \qquad (23)$$

The expected result that the RC decay should dominate at large $\tau$ appears explicitly.

Maximum Growth with z

If z is fixed and the real part of $g_s$ is maximum in $\tau$ we find a point of the class 2-type $\Delta = (\alpha\tau/Kz) < 8/(3\sqrt{3})$:

$$\mu = \frac{1}{\sqrt{8}} = .3535 ,$$

$$\psi = \pm\sqrt{\frac{3}{8}} = \pm .6124 ,$$

$$\frac{\alpha\tau}{Kz} = \frac{3}{4\sqrt{2}} = .5303 ,$$

$$R_e(g_s) = \frac{Kz}{2\sqrt{2}} = .3536\ Kz \ . \qquad (24)$$

This solution represents the peak of a wave packet moving backward in the pulse (increasing $\tau$) and forward in z with trajectory $\alpha\tau/Kz = .5303$. The same result is obtained by perturbing the beam sinusoidally at $z = 0$ for a long duration ($\tau$), with frequency $\omega_o = \alpha/\sqrt{3}$.

## Application to Heavy Ion Fusion Driver

The maximum growth is calculated here at a medium energy point in a fusion driver, with ion parameters (T = 1000 MeV, m = 200 amu, q = 1). We also adopt the previously given quantities ($C = 3 \times 10^{-10}$ F-m, $R = 300\Omega$/m, $I_o = 10^3$ A, $\tau_p$ = 500 ns). Then we have (non-relativistic calculation).

$$v_o = .104\ c, \quad \lambda_o = 32.2\ \mu C/m \ ,$$

$$\alpha^{-1} = 90.0\ ns \ , \quad K = 7.33 \times 10^{-3}\ m^{-1} ,$$

$$\alpha\tau = 5.56\ \tau/\tau p \ , \quad Kz = 7.33\ z_{km} ,$$

$$\Delta = \alpha\tau/Kz = .758\ \frac{\tau/\tau_p}{z_{km}} \ .$$

From eqs. (24) we have the maximum growth point for a perturbation initiated at the pulse head:

$$\frac{\tau/\tau_p}{z_{km}} = \frac{.5303}{.758} = .699 \ ,$$

$$\mathrm{Re}\ (g_s) = .3536\ Kz = 2.59\ z_{km} \ .$$

It is clear that this asymptotic limit is available within the 500 ns pulse length for z out to ~1.4 km, and several e-fold of growth can occur over this distance. The growth rate is small enough that it may be possible to control it with a feed-forward system. Note that this maximum growth rate, when associated with a perturbation of constant frequency at z = 0, occurs at

$$\nu_o = \frac{\alpha}{2\pi\sqrt{3}} = 1.02\ \mathrm{MHz} \ ,$$

which is very low considering the 500 ns pulse length. For the more reasonable $\nu_o$ = 10 MHz we find the very low growth rate with distance of .636 $km^{-1}$.

## References

1. Edward P. Lee, Proc. 1981 Linear Accelerator Conference, Los Alamos Report LA-9234-C, 263.