CUMULATIVE BEAM BREAKUP IN RADIO-FREQUENCY LINACS*

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Abstract

An analytic model of cumulative beam breakup has been developed which is applicable to both low-velocity ion and high-energy electron linear accelerators. The model includes arbitrary velocity, acceleration, focusing, initial conditions, beam-cavity resonances, and variable cavity geometry and spacing along the accelerator. The model involves a "continuum approximation" in which the transverse kicks in momentum imparted by the cavities are smoothed over the length of the linac. The resulting equation of transverse motion is solved via the WKBJ method. Specific examples are discussed which correspond to limiting cases of the solution.

Introduction

In the usual treatment of cumulative beam breakup (BBU), the cavities which comprise the linac are considered to have negligible length and to be the only source of deflecting fields. In addition, a "continuum approximation" is also commonly invoked in which the discrete kicks in transverse momentum imparted by the cavities are considered to be smoothed along the linac. With this approximation, the equation of transverse motion of the beam is:

$$\left[\frac{1}{\beta\gamma}\frac{d}{d\sigma}\left(\beta\gamma\frac{d}{d\sigma}\right) + (k_T\mathcal{L})^2\right]\xi(\sigma,\zeta) = \epsilon(\sigma)\int_0^\zeta d\zeta' w(\zeta-\zeta')F(\zeta')\xi(\sigma,\zeta') . \qquad (1)$$

Here, β and γ have their usual meanings; $\sigma=s/\mathcal{L}$ is a dimensionless spatial variable defined in terms of position along the linac, s, and the total length of the linac, $\mathcal{L}$; $\zeta=\omega(t-\int ds/\beta c)$ is the time, made dimensionless by use of the angular frequency ω of the deflecting mode, measured after the arrival of the head of the beam at s; k_T is the net transverse focusing wavenumber; ξ is the transverse displacement of the beam centroid from the axis; and $F(\zeta)=I(\zeta)/\langle I\rangle$ is the form factor for the current defined in terms of the beam current $I(\zeta)$ and average beam current $\langle I\rangle$. $\epsilon(\sigma)$, a dimensionless quantity which represents the strength of the BBU interaction, is given by

$$\epsilon(\sigma) = \frac{1}{2}\frac{\langle I\rangle Ze}{p}\frac{\Gamma}{\omega}\frac{\mathcal{L}^2}{L}, \qquad (2)$$

a product of quantities describing the beam, cavities, and linac in which Ze and p are the charge and momentum, respectively, of a constituent particle of the beam, and L is the spacing between neighboring cavities. $w(\zeta)$ is a dimensionless "wake function" given by

$$w(\zeta) = e^{-\zeta/2Q}\sin\zeta \quad for\ \zeta\geq 0; \quad = 0 \quad otherwise, \qquad (3)$$

in which Q is the quality factor of the deflecting mode under consideration. Γ, a geometry factor, is given by

$$\Gamma = \frac{2}{\epsilon_0\omega}\frac{\left|\int_0^\ell e^{-i\omega z/\beta c}\frac{\partial E_z(0,0,z)}{\partial x}dz\right|^2}{\int_V E^2(\mathbf{x})\,d\mathbf{x}}, \qquad (4)$$

in which the integrals are over the rf electric field of the deflecting mode, and ϵ_0 is the permittivity of free space.

Equation (1) is the foundation of the theory of cumulative beam breakup. Various forms of it have been solved analytically. Early investigators treated the case of a dc beam with no net transverse focusing both for $\beta=1$[1] and for $\beta\neq 1$[2], as well as the case of a focused relativistic dc beam.[3] These investigators developed approximate analytic solutions derived from a technique in which a Fourier or Laplace transform is applied to the equation of transverse motion, the transformed equation is integrated via the WKBJ method, and the transformation is inverted using the method of steepest descent. Later investigators applied this technique to the problem of relativistic, bunched electron beams. Both single-bunch[4] and multi-bunch[5-8] BBU have been considered, and the technique has been shown to give analytic results which agree very well with numerical integration where BBU is significant.[7,8] The analytic solution can be decomposed into a steady-state term valid after times long compared to Q/ω, and a transient term describing the approach to steady state. These features also apply to the case of a dc beam,[1,3,9] for which the exponential growth characterizing the transient behavior can also be predicted by assuming at the outset an oscillatory solution of the form $\exp(i\omega t-ikz)$.[10]

The principal qualitative distinction of the solution for a bunched beam versus that of a dc beam is the possibility for resonance between the deflecting-mode frequency and the bunch frequency. This resonance will determine whether the steady-state displacement grows exponentially along the linac. While the transient solution sets the most stringent constraint in high energy, short pulse length linacs such as linear colliders,[11] in cw linacs transient growth can be controlled,[12] and BBU behavior will then be set by the steady-state solution.

Beam breakup has yet to be investigated for a bunched beam of arbitrary β. This problem is important in the context of

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the development of cw, high-current ion linacs. An effort is underway to apply rf superconductivity to the development of these accelerators.[13] Because their constituent cavities will be short and independently phased, cumulative BBU is expected to be the dominant transverse instability, and it needs to be considered prior to the design of these linacs.

In this paper, we calculate the transverse behavior of a beam of arbitrary $\beta(\sigma)$ consisting of delta-function bunches in a linac with smoothly varying parameters. Two limiting cases are considered, one with no focusing and no acceleration, and the other with strong solenoidal focusing and a linear acceleration profile. The results for these cases are applied elsewhere to two conceptual low-β superconducting linacs.[13]

Beam Breakup with Delta-Function Bunches

With $x(\sigma,\zeta) \equiv \sqrt{\beta\gamma}\,\xi(\sigma,\zeta)$, eq. (1) becomes

$$\frac{d^2x(\sigma,\zeta)}{d\sigma^2} - [\phi(\sigma)-(k_T\mathcal{L})^2]x(\sigma,\zeta) = \epsilon(\sigma)\int_0^\zeta d\zeta' w(\zeta-\zeta')F(\zeta')x(\sigma,\zeta'), \quad (5)$$

where

$$\phi(\sigma) = \frac{2\gamma(\beta\gamma)^2(d^2\gamma/d\sigma^2)-(\gamma^2+2)(d\gamma/d\sigma)^2}{4(\beta\gamma)^4}. \quad (6)$$

We consider a beam consisting of delta-function bunches of identical charge separated by period τ. According to eq. (5), the displacement x_M of bunch M is governed by

$$\frac{d^2x_M(\sigma)}{d\sigma^2} - [\phi(\sigma)-(k_T\mathcal{L})^2]x_M(\sigma) = \epsilon(\sigma)\,\omega\tau\sum_{m=0}^{M-1} w_{M-m}\,x_m, \quad (7)$$

where $w_k = e^{-k\omega\tau/2Q}\sin k\omega\tau$. We solve this equation using the following Fourier series:[5]

$$X(\sigma,\theta) \equiv \sum_{m=0}^{\infty} x_m(\sigma)\,e^{im\theta};\quad x_M(\sigma) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta e^{-iM\theta}X(\sigma,\theta). \quad (8)$$

After transformation, eq. (7) governs $X(\sigma,\theta)$ in the manner:

$$\frac{d^2X(\sigma,\theta')}{d\sigma^2} - q^2(\sigma,\theta')X(\sigma,\theta') = 0, \quad (9)$$

where $\theta' \equiv \theta + i(\omega\tau/2Q)$, and

$$q^2(\sigma,\theta') = \frac{1}{2}\epsilon(\sigma)\frac{\omega\tau\sin\omega\tau}{\cos\theta'-\cos\omega\tau} + \phi(\sigma) - [k_T\mathcal{L}]^2. \quad (10)$$

In terms of the functions

$$\begin{Bmatrix} C(\sigma,\theta') \\ S(\sigma,\theta') \end{Bmatrix} = \sqrt{\frac{q(0,\theta')}{q(\sigma,\theta')}}\begin{Bmatrix}\cosh \\ \sinh\end{Bmatrix}\int_0^\sigma q(\sigma',\theta')\,d\sigma', \quad (11)$$

the WKBJ solution of eq. (9) is

$$X(\sigma,\theta) = C(\sigma,\theta')\sum_{m=0}^{\infty} x_m(0)\,e^{im\theta} + \frac{S(\sigma,\theta')}{q(0,\theta')}\sum_{m=0}^{\infty}\frac{dx_m(0)}{d\sigma}e^{im\theta}. \quad (12)$$

In turn, after some manipulation which is aided by noting that the Fourier series for $C(\sigma,\theta')$ and $S(\sigma,\theta')/q(0,\theta')$ contain only non-negative indices m, $x_M(\sigma)$ can be written in the form[14]

$$x_M(\sigma) = \frac{1}{2\pi}\sum_{m=0}^{M} e^{\frac{-m\omega\tau}{2Q}}\int_{-\pi}^{\pi} d\theta\, e^{-im\theta} * \left[x_{M-m}(0)\,C(\sigma,\theta) + \frac{dx_{M-m}(0)}{d\sigma}\frac{S(\sigma,\theta)}{q(0,\theta)}\right]. \quad (13)$$

Equation (13) allows for the incorporation of arbitrary initial conditions for the transverse motion of each beam bunch upon entry into the linac. In what follows, we shall consider only the case of a "misaligned beam" in which $\xi_M(0,0)=\xi_o$ and $d\xi_M(0,0)/d\sigma=0$ for every M. For this case, eq. (13) yields for the steady-state behavior ($M \to \infty$)

$$\xi_\infty(\sigma) = \xi_o\,\Xi(\sigma, i\omega\tau/2Q), \quad (14)$$

and for the approach to steady state ($M < \infty$),

$$\xi_M(\sigma) - \xi_\infty(\sigma) = -\frac{\xi_o}{2\pi}e^{-(M+1)\frac{\omega\tau}{2Q}}\int_{-\pi}^{\pi} d\theta\frac{e^{-iM\theta}\Xi(\sigma,\theta)}{e^{i\theta}-e^{-\omega\tau/2Q}}, \quad (15)$$

where

$$\Xi(\sigma,\theta) = \sqrt{\frac{\beta(0)\gamma(0)}{\beta(\sigma)\gamma(\sigma)}}\left[C(\sigma,\theta) + \frac{d\gamma(0)/d\sigma}{2\beta^2(0)\gamma(0)}\frac{S(\sigma,\theta)}{q(0,\theta)}\right]. \quad (16)$$

The integral in eq. (15) can be estimated by steepest descent. Two cases, the coasting beam with no focusing and the nonrelativistic accelerated beam with strong solenoidal focusing, will be considered. We now calculate the transverse displacement at the end of the linac ($\sigma=1$) for these cases.

Coasting Beam, No Focusing. For this case, β and γ are constants, and both ϕ and k_T are zero. If the linac is comprised of N identical, equally spaced cavities, then $\epsilon(\sigma)$ is also a constant. From eq. (14), the steady-state displacement is

$$\xi_\infty(N) = \xi_o\cosh\left[N\sqrt{\frac{\epsilon\omega\tau}{4}\left(\frac{L}{\mathcal{L}}\right)^2 p(\omega\tau)}\right], \quad (17)$$

in which $p(\omega\tau)$ includes the resonances between the frequencies $1/\tau$ of the accelerating mode and $\omega/2\pi$ of the deflecting mode:[6]

$$p(\omega\tau) = \frac{\sin\omega\tau}{\sinh^2\left(\frac{\omega\tau}{4Q}\right) + \sin^2\left(\frac{\omega\tau}{2}\right)}. \quad (18)$$

From eq. (15), steepest descent gives for the approach to steady state

$$\xi_N(N) - \xi_\infty(N) \approx -\xi_o \frac{\sqrt{E}\exp\left(\frac{3\sqrt{3}}{4}E - \frac{M\omega\tau}{2Q}\right)}{4M\sqrt{6\pi}\left[\sinh^2\left(\frac{\omega\tau}{4Q}\right) + \sin^2\left(\frac{\omega\tau}{2}\right)\right]} * \left[\left(\cos\omega\tau - e^{\frac{\omega\tau}{2Q}}\right)\cos\psi + \sin\omega\tau\sin\psi\right], \qquad (19)$$

where

$$E = \sqrt[3]{\epsilon\omega\tau\left(\frac{L}{\mathcal{L}}\right)^2 N^2 M}, \quad \text{and} \quad \psi = \frac{3}{4}E + \frac{\pi}{12} - M\omega\tau .$$

Accelerated Beam, Strong Solenoidal Focusing. For an accelerated beam with solenoidal focusing, the net focusing is, including radial defocusing from the accelerating gradient but ignoring defocusing due to space charge,

$$k_T^2 = \left(\frac{ZeB}{2p}\right)^2 + \frac{\pi\, ZeE_oT\sin\Phi}{pc\lambda_o(\beta\gamma)^2} = k_B^2 + k_E^2, \qquad (20)$$

where B is the magnetic field in the solenoids, $E_oT\cos\Phi$ is the real-estate accelerating gradient, and λ_o is the rf wavelength of the accelerating mode. We shall consider a nonrelativistic beam for which $\gamma \approx 1$ and which is being accelerated slowly enough and focused strongly enough that the first term in eq. (20) predominates. If B is sufficiently large and is uniform along the linac, then $k_T^2 \approx k_B^2 \gg \phi/\mathcal{L}^2$, and $k_T \propto 1/\beta$. Assuming the cavities have longitudinal dimensions scaling linearly with β and are spaced in the manner $L = L_o[\beta/\beta(0)] = 2\beta\lambda_o$, the geometry factor $\mathcal{L}$ is the same for each cavity, and therefore $\epsilon \propto 1/\beta^2$. With these assumptions, the steady-state displacement is

$$\xi_\infty(N) \approx \xi_o\cos[Nk_B(0)L_o], \qquad (21)$$

and the approach to steady state is

$$\xi_N(N) - \xi_\infty(N) \approx -\xi_o \frac{\sqrt{E}\exp\left(E - \frac{M\omega\tau}{2Q}\right)}{8M\sqrt{2\pi}\left[\sinh^2\left(\frac{\omega\tau}{4Q}\right) + \sin^2\left(\frac{\omega\tau}{2}\right)\right]} * \left[\left(\cos\omega\tau - e^{\frac{\omega\tau}{2Q}}\right)\cos\psi + \sin\omega\tau\sin\psi\right], \qquad (22)$$

where

$$E = \sqrt{\epsilon(0)\,\omega\tau\frac{L_o}{k_B(0)\,\mathcal{L}^2}NM}, \quad \text{and} \quad \psi = Nk_B(0)L_o - M\omega\tau .$$

Summary and Conclusions

Cumulative beam breakup in radio-frequency ion accelerators has been calculated using, as a model, a beam of delta-function bunches. As examples, the limiting cases of a coasting beam with no focusing and a nonrelativistic, slowly accelerated beam with strong solenoidal focusing were delineated. These cases are used in Ref. 13 for applications to conceptual designs of high-current ion accelerators.

Limits on the Q of the deflecting mode required to control BBU would be larger than that inferred from these calculations if a more realistic set of assumptions were used. For example, the cavities will carry a distribution of deflecting-mode frequencies due to construction tolerances, and this suppresses BBU.[15,16] In addition, the transient growth of BBU can be controlled by smoothly varying the charge per bunch.[12]

Beams in low-velocity linacs are comprised of bunches of nonzero length, a property not included here. Bunch length will be most important in the consideration of high-frequency cavity modes because a given bunch will then fill a large fraction of the rf period of the mode. This will, for example, modify the beam-cavity resonance. Using Fourier transforms rather than Fourier series to solve eq. (1), we have developed a formalism both to include nonzero bunch length and to calculate the behavior of test particles constituting a longitudinal halo between bunches. It will be discussed in future papers. The results reported here are accurate for deflecting modes of low frequency in which a given bunch fills a small fraction of the rf period of the mode.

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