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FORMULAS FOR THE TWISS BEAM PARAMETERS FOR A MATCHED BEAM AT THE ENTRANCE TO A LINAC*

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Abstract

Formulas for a transversely-matched beam are derived in terms of the r-matrix representation of accelerating and focusing elements in the first period of a linac structure. The formulas predict beam parameters close to the values actually used in tuning the LAMPF 805 MHz linac, and give reasonable-looking results in the other cases for which they have been applied.

Introduction

A beam occupying an ellipse in transverse phase-space may be said to be transversely matched into a linac when for a given emittance E, the Twiss parameters α and β at the start of the linac minimize the largest transverse widths of the beam in the first period or first few focusing periods of the linac. This results in a beam which is uniform (except for adiabatic damping) from one focusing period to the next: the width does not breathe in and out at twice the betatron frequency. We have found that we may obtain reasonably matched beams by finding α's and β's that are the same at the start and end of the first focusing period.

Transformation of α's and β's

The equation for the beam ellipse in one of the transverse phase spaces is of the form

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = E/\pi \,, \tag{1}$$

where x and x' are the position and divergence of a particle on the boundary of the ellipse, E is the emittance, and the Twiss parameters α, β, and γ are normalized such that

$$\beta\gamma - \alpha^2 = 1 \,. \tag{2}$$

The final coordinates x_f and x'_f at the end of a section of beam line may be related to the coordinates at the beginning using the R matrix:

$$\begin{bmatrix} x_f \\ x'_f \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \end{bmatrix} \,. \tag{3}$$

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For representing linacs, we need the transformation for the Twiss parameters through the section when the determinant of the R matrix is not unity. Using Eq. (1) and the inverse relation corresponding to Eq. (3), we find[1] the parameters at the end of the section are given in terms of those at the beginning by

$$\begin{bmatrix} \beta_f \\ \alpha_f \\ \gamma_f \end{bmatrix} = T \cdot \begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix} \,, \tag{4}$$

where

$$T = \frac{1}{d} \begin{bmatrix} r_{11}^2 & -2r_{11}r_{12} & r_{12}^2 \\ -r_{11}r_{21} & r_{11}r_{22}+r_{12}r_{21} & -r_{12}r_{22} \\ r_{21}^2 & -2r_{21}r_{22} & r_{22}^2 \end{bmatrix}$$

and $$d = r_{11}r_{22} - r_{12}r_{21} \,. \tag{5}$$

Phase Advance

We may also generalize the concept of phase advance to apply for situations when the determinant of the R matrix is not unity. Any matrix whose determinant is unity may be represented in the form

$$R_1 = \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{bmatrix} \tag{6}$$

for appropriate values of the parameters μ, α, β, and γ. The parameter μ is known as the phase advance. (We will come back to α, β, and γ in the following section.) If we have an R matrix for a focusing period of our structure for which the determinant d is not unity, the matrix

$$R_1 = \frac{1}{\sqrt{d}} \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \tag{7}$$

will have a unity determinant, and we can find μ from

$$\cos\mu = \frac{r_{11} + r_{22}}{2\sqrt{d}} \,. \tag{8}$$

For the types of linac that we have studied, μ increases as the focusing strength of the quadrupole doublets increases, and the focusing strength that gives the smallest beam size for a given emittance occurs near μ = 85 degrees. Thus μ may be used as a guide to find the optimum focusing strength.

Formulas for Matched α and β

We take the matched condition to be when the Twiss parameters for the beam ellipse are the same at the exit of the first focusing period of a linac as at the start. We assume we can find a value of μ (with $0 < \mu < \pi$) that satisfies Eq. (8). Then using Eqs. (6) and (7) we may take

$$\beta = \frac{r_{12}}{\sqrt{d}\ \sin\mu} , \tag{9}$$

$$\alpha = \frac{r_{11} - r_{22}}{2\sqrt{d}\ \sin\mu} , \tag{10}$$

and
$$\gamma = -\frac{r_{21}}{\sqrt{d}\ \sin\mu} . \tag{11}$$

(These satisfy the normalization indicated in Eq. (2), and thus only α and β are sufficient to specify a matched beam.) If we substitute the α, β, and γ given above in the right side of Eq. (4), we find $\alpha_f=\alpha$ and $\beta_f=\beta$, and thus Eqs. (9) - (11) give the Twiss parameters for the matched condition.

Application of the Formulas to the LAMPF 805-MHz Linac and a Pilac Design

The horizontal and vertical Twiss parameter values as used for normal operation of LAMPF 805-MHz linac and the values given by Eqs. (9) and (10) are listed in Table I. The calculated values are quite close to the standard ones used in practice, and a plot of the beam widths for the calculated match is practically indistinguishable from such a plot for the standard match.[1]

TABLE I
Standard and Calculated Parameters For The LAMPF 805-MHz Linac

	α_x (-)	β_x (cm/rad)	α_y (-)	β_y (cm/rad)
Std.	0.337	329.	1.240	556.
Calc.	0.402	329.	1.39	573.

The first focusing period of this linac takes protons from 100 to 103.2 MeV kinetic energy, resulting in an R matrix whose determinant has the value d=0.9836.

The preceding example had an R matrix for the first focusing period whose determinant was not far from unity. Next, we look at a case for which this determinant is significantly different from unity: a reference design for a linac for pions (pilac). The first focusing period of this linac uses nine cavities, and takes the pions from 500 to 634.5 MeV kinetic energy, resulting in an R matrix whose determinant has the value d=0.8198. The entire pilac has three focusing periods and takes the pions to 920 MeV. The widths of the beam for calculated values of the Twiss parameters are shown in Fig. 1, and each plane is reasonably matched. (There are nine lines on this plot per focusing period.)

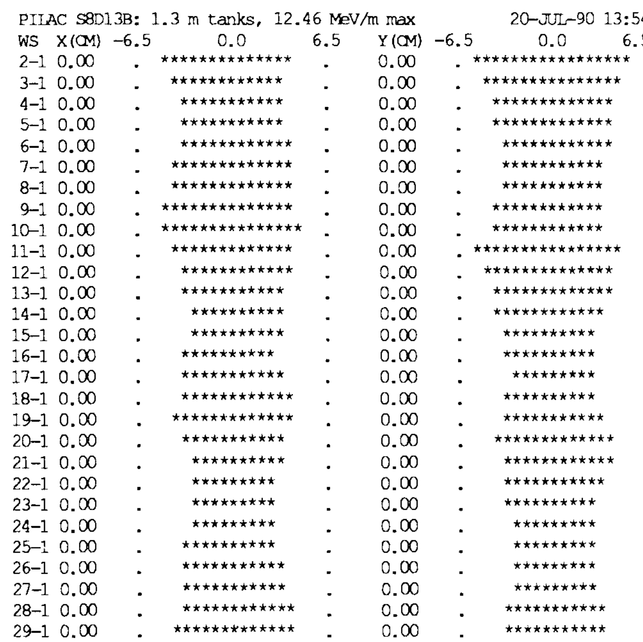

Fig. 1. Horizontal and vertical beam widths for a Pilac reference design, initial α's and β's from matching formulas.

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Reference

1. G. Swain, "Formulas for the Transverse Phase-Space Twiss Beam Parameters for a Matched Beam at the Entrance to a Linac," AHF Tech. Note 90-001, Los Alamos National Laboratory, Los Alamos, NM (Aug. 1990).