ANALYSIS OF COHERENT BEAM-BEAM KINK INSTABILITY IN A LINAC-ON-RING B FACTORY*

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Abstract

In this study, the dipole instability of the positron beam in a linac-on-ring collider is investigated as a multi-collision process. An integral equation for the vertical displacement of the positron beam in a beam-beam interaction is derived based on the cold fluid equations using the ribbon beam model[1, 2], where the force on the centroid of the election or positron bunch is approximated to be linear with their mutual offset. The integral equation is solved to the first order of iteration, assuming the phase shift for the positron bunch during each collision is small. Including only the linear betatron oscillation for the ring dynamics, the theory shows that in collisions with non-zero offset, the kink instability will cause a coherent growth of the transverse displacement of the positron beam. Simulation confirms this conclusion with good agreement.

Introduction

In a linac-on-ring B-factory, the high disruption of the electron beam affects the dynamics of the positron beam, making the stability of the positron beam in the ring an important issue for such a colliding scheme. In this study we have derived an integral equation for the vertical displacement of the positron beam in a linac-on-ring beam-beam interaction. The integral equation is based on the ribbon beam model and is solved to the first order of iteration, where we assume the phase shift for the positron bunch during collision is small ($\omega_p l_e/2 \ll 1$). This analysis reveals the coherent instability for the positron beam in a multi-collision process, which agrees well with the simulation results. Here only a linear matrix for betatron oscillation is considered in the ring.

Further studies showed that the transverse growth of the positron beam due to kink instability discussed in this paper can be suppressed when synchrotron motion is taken into account; the results will be presented in later works.

Equation of Motion for the Vertical Displacement of the Colliding Beams

For a charged beam, the dynamics of beam-beam interaction is governed by the cold fluid equations. Usually a beam is Gaussian in both transverse directions. We here study the case of a ribbon beam ($\sigma_x \gg \sigma_y$), where the beams are assumed to be uniform in the x direction so that the transverse degree of freedom is reduced to one. This model can serve to manifest the coherent beam-beam effect with relatively simpler analysis.

Let N_p be the total number of positrons in the positron bunch, and σ_{px}, σ_{py} and σ_{pz} be the rms bunch sizes in the x, y and z directions for the positron beam respectively. Also we denote $\bar{y}_p$ as the vertical displacement of the positron bunch at given z and t, and ρ_p as the line density function of the positron beam. The oscillation frequency for a single paraxial electron passing through the positron beam is ω_{e0}

$$\omega_{e0}^2 = \frac{2D_{ey}\rho_p}{\sigma_{pz}} \tag{1}$$

where D_{ey} is the electron disruption parameter

$$D_{ey} = \frac{2N_p r_0 \sigma_{pz}}{\gamma_e \sigma_{py}(\sigma_{px}+\sigma_{py})} \tag{2}$$

with r_0, the classical electron radius; and γ_e, the energy Lorentz factor for the electron beam. A set of analogous definition can be applied to the electron bunch by simply changing the subscript p to e.

In order to follow the dynamics of each beam, we transform the longitudinal coordinate z to the coordinate system z_e comoving with the electron bunch, and z_p comoving with the positron bunch. The vertical displacements of the two bunches are then

$$Y_e(z_e,t) = \bar{y}_e(z,t) \quad \text{and} \quad Y_p(z_p,t) = \bar{y}_p(z,t) \tag{3}$$

for $z_e = t - z$ and $z_p = t + z$. Here we set $c = 1$, so that the longitudinal velocity of the electron bunch is $u_{ez} = 1$, and the longitudinal velocity of the positron bunch is $u_{pz} = -1$.

Let l_e and l_p be the bunch lengths for the electron and positron bunch respectively. To the first order approximation, we assume that the average force on each bunch is linear with the offset of the two beams. As the consequence, we obtain a set of coupled differential equations of motion for the vertical displacement of the longitudinally uniform beams:

$$\begin{cases} \dfrac{\partial^2 Y_e(z_e,t)}{\partial t^2} + \omega_e^2 Y_e(z_e,t) = \omega_e^2 Y_p(2t - z_e, t) \\ \qquad (\dfrac{z_e}{2} \le t \le \dfrac{z_e + l_p}{2}) \\ \dfrac{\partial^2 Y_e(z_e,t)}{\partial t^2} = 0 \\ \qquad (0 \le t \le \dfrac{z_e}{2} \quad \text{and} \quad \dfrac{l_p + z_e}{2} \le t \le \dfrac{l_e + l_p}{2}) \end{cases} \tag{4}$$

for $0 \le z_e \le l_e$, and

$$\begin{cases} \dfrac{\partial^2 Y_p(z_p,t)}{\partial t^2} + \omega_p^2 Y_p(z_p,t) = \omega_p^2 Y_e(2t - z_p, t) \\ \qquad (\dfrac{z_p}{2} \le t \le \dfrac{z_p + l_e}{2}) \\ \dfrac{\partial^2 Y_p(z_p,t)}{\partial t^2} = 0 \\ \qquad (0 \le t \le \dfrac{z_p}{2} \quad \text{and} \quad \dfrac{l_e + z_p}{2} \le t \le \dfrac{l_e + l_p}{2}) \end{cases} \tag{5}$$

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for $0 \le z_p \le l_p$, with

$$\omega_e^2 = \frac{2f_e D_e}{\sigma_{pz} l_p} \quad \text{and} \quad \omega_p^2 = \frac{2 f_p D_p(z_p)}{\sigma_{ez} l_e}. \tag{6}$$

Here f_e and f_p are correction factors related to both the electron and positron distributions, which are taken to be constants in this study. Equations (4) and (5) are the fundamental tools for solving the behavior of the positron bunch in the beam-beam interaction.

Integral Equation for the Positron Beam and Its First Order Solution

We first proceed to solve the equation of motion by shifting the origin of time for any given slice of the electron beam to the moment when the positron bunch starts to pass through that slice. Define $\tau_e = t - \frac{z_e}{2}$, and

$$Y_e^{(e)}(z_e, \tau_e) = Y_e(z_e, t); \quad Y_p^{(e)}(z_p, \tau_e) = Y_p(z_p, t). \tag{7}$$

Then Eq.(4) for $z_e/2 \le t \le (z_e + l_p)/2$ can be written as

$$\frac{\partial^2 Y_e^{(e)}(z_e, \tau_e)}{\partial \tau_e^2} + \omega_e^2 Y_e^{(e)}(z_e, \tau_e) = \omega_e^2 Y_p^{(e)}(2\tau_e, \tau_e + \frac{z_e}{2}) \tag{8}$$

for $0 \le \tau_e \le \frac{l_p}{2}$. Solving the above eqation by Laplace transformation on the τ_e variable, we can express $Y_e^{(e)}(z_e, \tau_e)$ in terms of its initial conditions and $Y_p^{(e)}(z_p, \tau_e)$. Analogously, we define $\tau_p = t - \frac{z_p}{2}$, and

$$Y_p^{(p)}(z_p, \tau_p) = Y_p(z_p, t); \quad Y_e^{(p)}(z_e, \tau_p) = Y_e(z_e, t). \tag{9}$$

The same scheme of Laplace transform is also applied to Eq.(5). This leads to a coupled integral equation, which can be combined into one integral equation for the positron beam. For a constant initial offset y_0 of the electron beam, we have $Y_e^{(e)}(z_e, 0) = y_0$ and $Y_e^{(e)'}(z_e, 0) = 0$. The ultimate integral equation for $Y_p^{(p)}(z_p, \tau_p)$ is then

$$\begin{aligned} Y_p^{(p)}(z_p, \tau_p) &= Y_p^{(0)}(z_p, \tau_p) \\ &+ \omega_p \left(\frac{\omega_e}{2}\right) \int_0^{\tau_p} d\tau_p' \sin \omega_p(\tau_p - \tau_p') \\ &\times \int_0^{z_p} dz_p' \sin \frac{\omega_e(z_p - z_p')}{2} Y_p^{(p)}(z_p', \tau_p') \\ &(0 \le z_p \le l_p, \quad 0 \le \tau_p \le l_e/2), \end{aligned} \tag{10}$$

where $Y_p^{(0)}(z_p, \tau_p)$ represents the initial condition of the positron beam and the effect of initial electron offset on the positron beam:

$$\begin{aligned} Y_p^{(0)}(z_p, \tau_p) &= Y_p^{(p)'}(z_p, 0) \frac{\sin \omega_p \tau_p}{\omega_p} + Y_p^{(p)}(z_p, 0) \cos \omega_p \tau_p \\ &+ y_0 \cos(\frac{\omega_e z_p}{2}) (1 - \cos \omega_p \tau_p). \end{aligned} \tag{11}$$

In Eq.(10) the integral over z_p' is related to the evolution of electrons due to the passage of positrons, and the integral over τ_p' is related to the evolution of positrons due to their interaction with the electron beam.

The integral equation can be solved to the first order iteration when $(\omega_p l_e/2)^2 \ll 1$, corresponding to the case when the phase shift of the positron beam in each collision is a small number. This yields

$$\begin{aligned} Y_p^{(p)}(z_p, \tau_p) &= Y_p^{(0)}(z_p, \tau_p) \\ &+ \omega_p \left(\frac{\omega_e}{2}\right) \int_0^{\tau_p} d\tau_p' \sin \omega_p(\tau_p - \tau_p') \\ &\times \int_0^{z_p} dz_p' \sin \frac{\omega_e(z_p - z_p')}{2} Y_p^{(0)}(z_p', \tau_p') \\ &(0 \le z_p \le l_p, \quad 0 \le \tau_p \le l_e/2). \end{aligned} \tag{12}$$

Denoting as X_0 the vectors for the initial positron state at $t = 0$, and as X_f the vectors for the final positron state at the end of of first collision at $t = t_f = (l_e + l_p)/2$,

$$X_f = \begin{bmatrix} Y_p(z_p, t_f) \\ Y_p'(z_p, t_f) \end{bmatrix}, \quad \text{and} \quad X_0 = \begin{bmatrix} Y_p(z_p, 0) \\ Y_p'(z_p, 0) \end{bmatrix}. \tag{13}$$

We are able to write the final state of the vertical displacement for the positron beam at $t = t_f$ in terms of its initial state at $t = 0$ according to Eq.(12):

$$X_f = M^{(B)} X_0 + A + \lambda O(X_0) + \lambda P(A). \tag{14}$$

The matrix $M^{(B)}$ represents the linear betatron oscillation matrix due to beam-beam interaction along with the drift before and after the interaction. In the case when $\omega_p l_e/2 \ll 1$, we have

$$M^{(B)} \approx \begin{pmatrix} \cos \frac{\omega_p l_e}{2} & \frac{1}{\omega_p} \sin \frac{\omega_p l_e}{2} \\ -\omega_p \sin \frac{\omega_p l_e}{2} & \cos \frac{\omega_p l_e}{2} \end{pmatrix} \tag{15}$$

The vector A is defined to describe the effect of the initial electron offset on the positron beam

$$A = \begin{pmatrix} y_0 \cos \frac{\omega_e z_p}{2} \left[1 - \cos \frac{\omega_p l_e}{2} + \frac{\omega_p (l_p - z_p)}{2} \sin \frac{\omega_p l_e}{2}\right] \\ y_0 \omega_p \cos \frac{\omega_e z_p}{2} \sin \frac{\omega_p l_e}{2} \end{pmatrix} \tag{16}$$

The operators O and P are actually matrices with their elements being integral operators:

$$\begin{aligned} \lambda O(X_0) &= \int_0^{l_e/2} d\tau_p' \omega_p G(l_e/2 - \tau_p') \\ &\times \int_0^{z_p} dz_p' \frac{\omega_e}{2} \sin \frac{\omega_e (z_p - z_p')}{2} F_1(z_p', \tau_p'), \end{aligned} \tag{17}$$

$$\begin{aligned} \lambda P(A) &= \int_0^{l_e/2} d\tau_p' \omega_p G(l_e/2 - \tau_p') \\ &\times \int_0^{z_p} dz_p' \frac{\omega_e}{2} \sin \frac{\omega_e (z_p - z_p')}{2} F_2(z_p', \tau_p'), \end{aligned} \tag{18}$$

where the vector $G(\tau_p)$ in the above integrands is

$$G(\tau_p) = \begin{pmatrix} \sin \omega_p \tau_p + \frac{\omega_p (l_p - z_p)}{2} \cos \omega_p \tau_p \\ \omega_p \cos \omega_p \tau_p \end{pmatrix}, \tag{19}$$

and the functions $F_1(z'_p, \tau'_p)$ and $F_2(z'_p, \tau'_p)$ are

$$F_1(z'_p, \tau'_p) = Y'_p(z'_p, 0)\frac{\sin \omega_p \tau'_p}{\omega_p} + [Y_p(z'_p, 0) + \frac{z'_p}{2} Y'_p(z'_p, 0)] \cos \omega_p \tau'_p, \quad (20)$$

$$F_2(z'_p, \tau'_p) = y_0 \cos(\frac{\omega_e z'_p}{2}) (1 - \cos \omega_p \tau'_p). \quad (21)$$

Equation (14) makes it possible for us to study the positron beam instability in a multi-collision process. Applying Eq.(14) N times, and keeping only the first order of λ, we obtain a general expression of X_N for the positron state before the $(N+1)$-th collision in terms of the initial pre-collision state X_0, which describes how the positron bunch evolves with the number of revolutions in the ring.

As a simple example, the linear matrix in the ring is chosen to be a unit matrix. Also we set the average initial vertical displacement and momentum for the positron beam to be zero; i.e., $X_0 = 0$. A detailed calculation shows

$$Y_p(z_p, t_N) = -y_0 N \left(\frac{\omega_p l_e}{4}\right) \left(\frac{\omega_p z_p}{4}\right) \times \sin \left[\frac{\omega_p l_e}{2} \left(N - \frac{3}{2}\right)\right] \sin \frac{\omega_e z_p}{2}. \quad (22)$$

This equation exhibits the character of the motion for the vertical displacement of the positron beam, which performs sinusoidal oscillations in both space and time, with the amplitude proportional to both the number of collisions N and the longitudinal distance from the head of the beam z_p.

Comparison with Simulation Results

The simulation code SWARM[3] is used here for comparison of analytical and numerical results. The recirculation of the positron beam through the ring by a linear betatron matrix is implemented in the code. The parameters for the two beams used in the simulation are listed in Table 1, where the disruption parameters are calculated by approximately writing $\sigma_{ez} \approx l_e/4$ and $\sigma_{pz} \approx l_p/4$.

In the code each bunch, simulated by 2000 macroparticles in 50 slices, has a Gaussian distribution in the transverse plane and a uniform distribution longitudinally along the bunch length. For the 30th slice from the head of the positron bunch(i.e., $z_p/l_p = 0.6$), the plot for Y_p vs. the number of collisions N is shown as the dotted curve in Fig.1. At the pre-collision state of the 150th collision, we also plot Y_p against z_p, as shown with the dotted curve in Fig.2. By properly choosing the parameters in Eq.(6) to be $f_e = 0.9$ and $f_p = 0.47$, we get $\omega_e l_p/2 = 17$ and $\omega_p l_e/2 = 0.1$. Eq.(22) can then be used to calculate the analytic solution of $y_p(0.6 l_p, t_N)$ as a function of N, which gives the solid curve in Fig.1. Similarly, the solid curve in Fig.2 was obtained from Eq.(22) with N=150. The good agreement between the two curves in Fig.1 indicates that the analytic ribbon beam model indeed reveals the mechanism of instability shown in the simulation.

The analysis in Eq.(22) predicts that the instability will keep the vertical displacement for the positron beam growing in proportion to the collision number. However, in the simulation we observed that this growth is saturated after a certain N. It is believed that nonlinear effects are responsible for this phenomenon.

Acknowledgments

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References

[1] Y. Chin, DESY 87-011, 1987.

[2] K. Yokoya and P. Chen, Lecture Notes in Physics 400, Springer-Verlag, 1990.

[3] J. R. Boyce, S. Heifets and G. A. Krafft, 1990 Linac Conference, Albuquerque, NM, September, 1990.

TABLE 1
Parameter List in Simulation

e^-	e^+	Collision
$E_e = 2$ GeV	$E_p = 10$ GeV	$D_{ex} = 18$
$N_e = 0.5 \times 10^9$	$N_p = 10^{12}$	$D_{ey} = 180$
$l_e = 500\ \mu$m	$l_p = 500\ \mu$m	$y_0 = y_{\text{offset}} = 0.1\ \mu$m
$\sigma_{ey} = 0.3\ \mu$m	$\sigma_{py} = 0.3\ \mu$m	
$\sigma_{ex} = 3\ \mu$m	$\sigma_{px} = 3\ \mu$m	

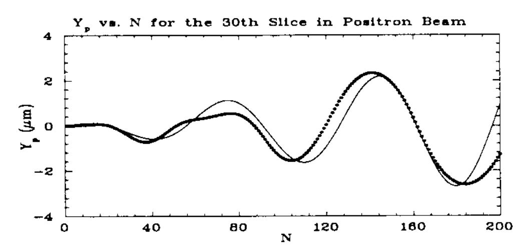

Figure 1: Comparison of analytical results (solid curve) with numerical results (dotted curve) for Y_p vs. collision number N with $z_p/l_p = 0.6$.

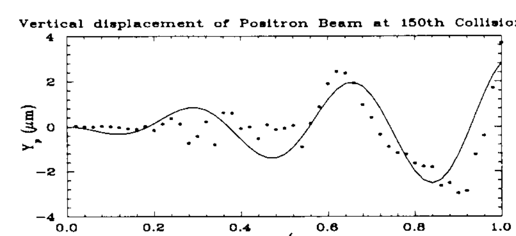

Figure 2: Comparison of analytical results (solid curve) with numerical results (dotted curve) for Y_p vs. the longitudinal distance z_p at the pre-collision state for $N = 150$.