FREQUENCY SPECTRA FROM SOLENOID LATTICE ORBITS*

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Abstract

Multi-charge state heavy ion beams have been proposed to increase average beam intensity in rare isotope drive linacs [1]. However, the dynamics of multi-charge state beams make it challenging to optimize the beam quality in low energy linacs. One of the primary complications is that the multiple charge states introduce different focusing effects in the beam dynamics. This leads to a large frequency spectrum in the transverse motion of the beam centroid. Matlab simulations are used to describe how the frequency spectrum of the centroid transforms when the reference charge state is changed in accelerating, space charge free solenoid lattices. These frequency shifts can then be used to predict the behavior of beam of known composition using the frequency spectrum of BPM signals.

INTRODUCTION

Multicharge state beams have been proposed to increase beam intensity in ion accelerators [1]. While the dynamics of individual charge states are understood, there is no formalism for analysing the orbits of multicharge state beams. Beam osition monitor (BPM) measurements of multicharge state beams look only at the center of the beam, not the distribution of beam. Therefore the individual charge states cannot be resolved from these measurements and the center will be the weighted average of each center of the individual charge states.

The orbit of each charge state in the beam is straightforward to compute. For a magnetic quadrupole lattice, where the x and y dynamics are uncoupled, the multicharge state center evolution can be simulated. However, in a solenoid lattice the x-y coupling makes the analysis more difficult. The problem can be simplified using the Larmor frame. However, this requires the solutions of each charge state to be transformed into the lab frame. The orbits in the lab frame are rather complex. Therefore, instead of trying to average multiple positions, it is simpler to view the solution in the frequency domain. In the frequency domain, the multicharge state orbits can be represented by the superposition of the frequency spectra from each charge state. Also, rotations into different frames turn into splitting of peaks by adding and subtracting the rotation frequency from the original peak.

Therefore, by understanding the how the frequency spectra of the positions of each charge state evolve with the lattice parameters, it is possible to formulate a model to predict the behavior of the centroid of a multicharge state beam.
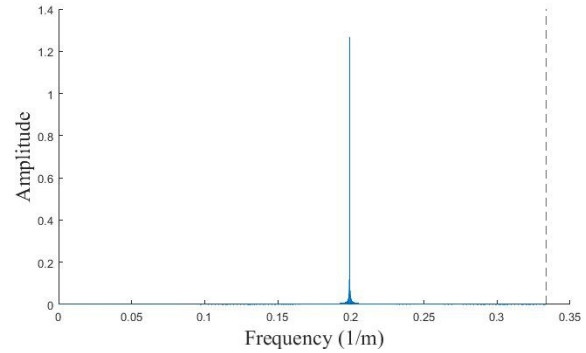
Figure 1: Frequency Spectrum from Hill's equation with κ=4, η=1/3, L = 1.5 and cos(σ)=0.3. The dashed line is at 1/(2L).

MAPPING SOLUTION

A simple method for generating single particle beam obits in a periodic, hard edge, solenoid beam line is to use the transfer matrix, M. For a solenoid lattice of length L, focusing strength κ, and occupancy η the x offset and velocity in the Larmor frame is given by the transfer matrix [2]:

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{s=(n+1)L} = M \begin{bmatrix} x \\ x' \end{bmatrix}_{s=nL} \tag{1}$$

where

$$M = \begin{bmatrix} \cos(\sqrt{\kappa}L\eta) - \sqrt{\kappa}L(1-\eta)\sin(\sqrt{\kappa}L\eta) \\ -\sqrt{\kappa}\sin(\sqrt{\kappa}L\eta) \\ \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}L\eta) - L(1-\eta)\cos(\sqrt{\kappa}L\eta) \\ \cos(\sqrt{\kappa}L\eta) \end{bmatrix} \tag{2}$$

$$\kappa = \left(\frac{qB}{2m\gamma\beta c}\right)^2 \tag{3}$$

where q is the charge of the particles, B is the magnetic field of the solenoid, m is the mass of the particles, ß is the normalized velocity, and γ is the relativistic factor.

The transfer matrix can generate beam orbits for long, periodic beam lines very quickly. Because of the fixed step size a discrete Fourier transform was used to determine the frequency spectrum of the positions (Fig 1). The transform was calculated using the fast Fourier transform (FFT) function in Matlab [3].

However, the step size is fixed at the lattice period length. This large step size causes aliasing which must be accounted for. The aliasing causes all frequencies to be mapped to a frequency between $-\frac{1}{2L}$ and $\frac{1}{2L}$. The aliased spectrum was found to only include none peak. Therefore

all the frequency peaks must exist at $\frac{n}{2L} \pm \Delta f$ for some integer n and some fixed Δf. Therefore by only knowing the single aliased peak, one can predict possible locations of the peaks in the frequency spectrum. However, it is not guaranteed that all predicted locations will have a peak.

Phase Advance

The phase advance per lattice period, σ, of a hard edge solenoid FOFO lattice in the Larmor frame as a function of the period length, L, occupancy, η, and focusing strength, κ, can easily be calculated by half the trace of the transfer matrix [2].

$$\cos(\sigma) = \cos(\sqrt{\kappa}L\eta) - \frac{1}{2}\sqrt{\kappa}L(1 - \eta)\sin(\sqrt{\kappa}L\eta) \quad (4)$$

If the lattice period has length L then the spatial frequency of the mapping is given by:

$$f(\sigma) = \frac{\sigma}{2\pi L} \quad (5)$$

This frequency exactly correspondences to the frequency peak found from the transfer map and therefore also indicates the possible locations of all frequency peaks.

HILL'S EQUATION SOLUTION

The beam offsets for the solenoid FOFO lattice were also determined using a fourth order Runge Kutta solver to solve Hill's equation in the Larmor frame [2].

$$x'' + \kappa x = 0 \quad (6)$$

This method was chosen over the methods contained in Matlab for several reasons. Firstly, the orbits needed to be calculated at regular intervals in order to take a discrete Fourier transform. This is simply done with a written fixed step size solver while Matlab differential equation solvers use variable step sizes. It is possible to select points at regular intervals after integration, however no benefit was seen for using this method. Another issue with the Matlab solvers is Hill's equation is a second order differential equation and the Matlab solvers are first order solvers. The form of Runge Kutta for second order differential equations is known and simple to program. Determining the proper method to use the Matlab solvers for these types of problems was deemed to be more difficult than simply writing down a known form.

The primary disadvantage of the Runge Kutta solver is it is not symplectic. Therefore, it is expected that the result will drift away from the actual solution. To determine if this was an issue, the Courant-Snyder invariant was calculated before and after integration. The results showed a change on the order of 10^{-12} in the invariant, so the non-symplectic nature of Runge Kutta is not an issue.

The resulting spectra consisted of a series of peaks located at $\frac{n}{2L} \pm \Delta f$ for all n with the lowest peak located at $\frac{\sigma}{2\pi L}$ (see Fig. 2). The mapping solution is therefore sufficient to determine the location of all frequency peaks because no peaks are missing.

The symmetric behavior of the frequency peaks lead to concerns that the FFT was showing multiple Nyquist zones instead of real signals. To check this, the orbits, x(s), were

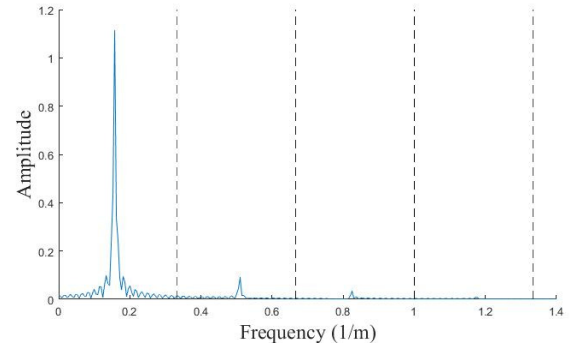multiplied by the cosine of the frequency at each peak then integrated over the lattice:

Figure 2: Frequency Spectrum from Hill's equation with κ=4, η=1/3, and L = 1.5. The dashed lines are at n/(2L).
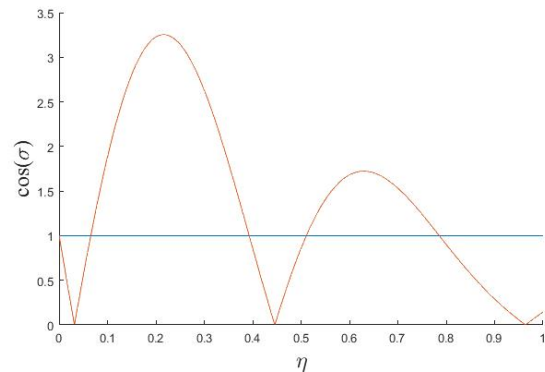

Figure 3: Instability criterion for varying η with $\sqrt{\kappa}L = 4$.

$$\int x(s)\cos(2\pi f_n s)ds \quad (7)$$

where f_n is the nth peak in the frequency spectrum. This integral will give a value close to zero if the signal is not real and a larger value if there is a signal at f_n. These integrals showed that there are definite signals at all the peaks in the spectrum and nowhere else.

The phase advance analysis not only tells where the peaks are, but can also predict the frequency peak with the largest amplitude using the instability criteria, $|\cos(\sigma)| \leq 1$ [2]. By varying κ, η, and L one can plot $\cos(\sigma)$ as a function of these three variables (see Fig. 3). The resulting curves are a series of bumps that sometimes go over one. In the regions where the cosine of the phase advance is greater than one, the chosen parameters lead to an unstable beam. If the parameters are chosen such that they lie before the first unstable region then the first peak in the frequency spectrum will be the dominant peak. If they lie between the first and second peak, then the second peak will be dominant, and so on. This gives a simple criterion for the main frequency component of the spectrum. This behavior suggests that when an unstable region is crossed, the branch of $\cos(\sigma)$ is changed by 2π. However, this method does not give relative amplitudes for the rest of the peaks.

Multicharge State Spectra

The spectrum of a multicharge state beam will be a superposition of the spectra of individual charge states. The charge dependence is contained in the focusing strength, see equation 3, therefore understanding how the frequency peaks vary with κ is equivalent to understanding how the peaks vary with the charge. One method of determining how the frequencies will change with different charges is to use the phase advance per lattice period (Eq. 4) which depends on the focusing strength. In the region below $\frac{1}{2L}$, as κ increases, the frequency peak shifts upwards. Each consecutive region changes the direction that the peak shifts, i.e. a downward shift in the first region would cause an upward shift to the peak in the second region, a downwards shift in the third region, etc. This behavior was expected due to the branch nature of σ.

The frequency shift is the same in each region due to the symmetry about $\frac{n}{2L}$. Therefore, for each charge state, only the lowest shift needs to be calculated using equation 5 to determine the shifts of all the peaks.

Coordinate Transforms

An understanding of the frequency spectra in the Larmor frame is a simple starting point. However, each charge state in a multicharge state beam will have its own Larmor frame that is rotating at a different frequency from the rest of the charge states. In order to construct the spectrum of a multicharge state beam it is necessary to transform the component spectra into a common frame. A simple frame to use is the lab frame. To transform into this frame, the Larmor rotation must be taken out of the positions. In the continuum solution, this is difficult because the Larmor frame is not continuously rotating. In order to change frames, the Larmor phase must be calculated at each point and the transform is done in the time domain before the FFT is taken. However, in the mapping solution, the Larmor phase advance is the same between each point and can simply be taking out by multiplying by a sine and cosine of the negative Larmor frequency times the position. Simple trigonometry shows that a frequency peak will be split into two peaks: the original peak plus and minus the Larmor frequency.

$$\cos(2\pi f s)\sum_n \cos(2\pi f_n s) =$$
$$\sum_n [\cos(2\pi(f - f_n)s) + \cos(2\pi(f + f_n)s)] \qquad (8)$$

While, this is obviously true for the single peak generated by the mapping, some computational tests have shown that this also holds for all peaks in the Hill's equation solution. Therefore, despite not accounting for the discontinuous rotation, only the mapping is necessary to determine the entire spectrum in any frame.

Once all the spectra have been rotated into a common frame, they can be superimposed to give the full frequency spectrum of the multicharge state beam. The same method can be applied to put the solution in any frame that rotates by the same phase each lattice period.

HILL'S EQUATION SOLUTION

So far, the analysis has only been concerning hard-edge solenoids and non-accelerating beam lines. These are not accurate descriptions of real accelerators. Accelerating dynamics can easily be added using Hill's equation. To simplify the dynamics, transit time factor formalization was used in the accelerating gap. The dynamics were also simplified by changing the field of the solenoids with the increase in velocity to keep the focusing strength the same across the entire lattice, i.e. $\frac{B}{\gamma\beta}$ was kept fixed. When integrated, no change in the frequency spectrum from the non-accelerating case was found. This was expected because the focusing strength in the accelerating gap was much smaller than the focusing strength of the solenoid.

Non-hard edge magnetic field profiles have been used to view the effects of fringe fields. The most basic case was to use the on axis field of a thin solenoid to determine the strength of the magnetic field. This field description is easy to use because it is analytic, making it easy to implement, and it is easy to change the shape of the field by adjusting the length and radius. The frequency spectrum of lattices containing the fringe fields were compared to the equivalent hard-edge models that can be found from integrating the magnetic field and the magnetic field squared. The resulting spectra showed some shifts from the hard edge equivalent models. The nature of the shifts has not been determined.

CONCLUSIONS

Using these methods, the frequency spectrum a single charge state in a hard-edge, periodic solenoid lattice can be calculated and transformed into any rotating frame. This allows for multi-charge state frequency spectra to be easily calculated by superpositions of the component spectra.

This analysis has been done with a very simplified lattice: hard edge, non-accelerating, and periodic. This is far from a good approximation of a realistic beam line. However, the methods described here can be applicable to more complex lattices and further work into these is currently being done.

REFERENCES

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