USING SQUARE MATRIX TO REALIZE PHASE SPACE MANIPULATION AND DYNAMIC APERTURE OPTIMIZATION*

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Abstract

A new method of using linear algebra technique to analyze periodical nonlinear beam dynamics is presented in ref. [1]. For a given system, a square upper triangular nonlinear transfer matrix is constructed out of the truncated power series transfer map. The square matrix is first separated into different invariant subspaces with much lower dimensions and we only focus on few invariant subspaces. An excellent action-angle approximation to the solution of the nonlinear dynamics can be obtained after applying Jordan transformation. We found that the deviation of linear action-angle invariant (i.e. Courant-Snyder invariant) from constancy of the new action provides a measure of the nonlinear of the motions. Therefore the square matrix provides a novel method to optimize the nonlinear dynamic system, and manipulate phase space as well. A chromaticity +7/+7 lattice of the NSLS-II optimized with this method was successfully commissioned. Our studies show that a basic "principle" – confining tune-shift-with-amplitude to prevent tune from crossing resonances in designing strong focusing storage rings, with which was complied by accelerator physicists for several decades, may not be an absolutely necessary condition.

INTRODUCTION

The question of the long term nonlinear behavior of charged particles in storage rings has a long history. To gain understanding, one would like to analyze particle motion under many iterations of the one turn map. The most reliable numerical approach is the use of a tracking code with appropriate local integration methods. For analysis, however, one would like a more compact and efficient representation of the one-turn-map out of which to extract relevant information. Among the many approaches to this issue we may mention canonical perturbation theory, Lie operators, power series, and normal form etc. Here, we would like to look at this problem from a somewhat different perspective, i.e., using linear algebra technique to analyze and optimize nonlinear beam dynamics. The theory on the square matrix method is explained in ref. [1]. In the following section, we only summarize this method briefly.

THEORY

For a given periodical nonlinear dynamic system, such as a particle moving in a storage ring, we use the complex Courant-Snyder variable $z = \bar{x} - i\bar{p}$, its conjugate $z^* = \bar{x} + i\bar{p}$ and powers to form a vector $\mathbf{Z} = (1, z, z^*, z^2, zz^*, \cdots, z^{*n})^T$, where ^{T} means taking the transpose of a vector. Here we use 1D motion to simplify the notation. The one turn map to transfer an initial status $\mathbf{Z}_0$ to its final status $\mathbf{Z}_1$ can be represented by a square matrix $\mathbf{M}$:

$$\mathbf{Z}_1 = \mathbf{M}\mathbf{Z}_0 \tag{1}$$

The matrix $\mathbf{M}$ is upper-triangular, and has the form

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \mathbf{M}_{11} & \cdots & \mathbf{M}_{1n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{M}_{nn} \end{pmatrix} \tag{2}$$

Here each submatrix $\mathbf{M}_{ij}$ has different dimensions respectively. Among them, $\mathbf{M}_{ii}$'s are square diagonal submatrices. A great simplification comes from a fact that the matrix is upper tridiagonal with all its eigenvalues given by its diagonal elements precisely determined by the tune, which represents the oscillation's phase advance per turn $\mu = 2\pi\nu$ solely. We can separate the full space spanned by the matrix columns into different invariant subspaces according to the eigenvalues. For example, all $z(zz^*)^k, k = 0, 1, \cdots$ belong to a same invariant space of the eigenvalue $e^{i\mu}$. We found that the simplest invariant subspaces $e^{i\mu}$ already provides a wealth information about dynamics. In this way, the high dimension matrix is reduced to several much lower dimension ones. For example, for a 2D $x - y$ system (4D in phase space), if we truncate the square matrix up to the 7^{th} order, its dimension is 330×330. The reduced matrix dimension for $e^{i\mu_x}$ and $e^{i\mu_y}$ is only 10×10 respectively. Then a stable Jordan decomposition, can be obtained on the low dimension submatrices $\mathbf{M}_j$

$$\mathbf{N}_j = \mathbf{U}_j\mathbf{M}_j\mathbf{U}_j^{-1} = e^{i\mu_j\mathbf{I}_j + \tau_j} \tag{3}$$

where the matrix $\mathbf{N}_j$ with $j = 1, 2, \ldots$ is the Jordan block with eigenvalue $e^{i}\mu_j$, corresponding to the j^{th} invariant subspace inside the space of vector $\mathbf{Z}$. $\mathbf{I}_j$ is the identity matrix, while τ_j is a superdiagonal matrix:

$$\tau_j = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{4}$$

The transfer of $\mathbf{Z}_0$ by the one turn map $\mathbf{M}$ inside the j^{th} subspace can be re-written as

$$\mathbf{W}_1 = \mathbf{U}\mathbf{Z}_1 = \mathbf{U}\mathbf{M}\mathbf{Z}_0 = e^{i\mu\mathbf{I}+\tau}\mathbf{W}_0 \tag{5}$$

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Unless explicitly stated, otherwise the subscript $_j$ is dropped off since now. KAM theory states that the invariant tori are stable under small perturbation. For sufficiently small amplitude of oscillation in $\mathbf{Z}$, the invariant tori are deformed and survive, i.e., the motion is quasi-periodic. So the system has a nearly stable frequency, and when the amplitude is small, the fluctuation of the frequency is also small. Thus for a specific initial condition described by $\mathbf{Z}_0$, the rotation in the eigenspace should be represented by a phase factor $e^{i(\mu+\phi)}$ so that

$$\mathbf{W}_1 = e^{i\mu\mathbf{I}+\tau}\mathbf{W}_0 \cong e^{i(\mu+\phi)}\mathbf{W}_0. \quad (6)$$

τ in Eq. (4) has no proper eigenvector, but only generalized eigenvectors. However, as the dimension of the eigenspace increases and approaches infinity, the eigenvector of τ is defined as a coherent state:

$$\tau\mathbf{W}_0 \cong i\phi\mathbf{W}_0. \quad (7)$$

Re-write $\mathbf{W}_0$ as a column with

$$\mathbf{W}_0^T = (w_0, w_1, w_2, \ldots) \quad (8)$$

Here $w_j = r_j e^{i\theta_j}$ are the new action-angle variables. The polynomials in Eq. (8) are $w_0 = u_0\mathbf{Z}_0, w_1 = u_1\mathbf{Z}_0, w_2 = u_2\mathbf{Z}_0, \cdots$. Then Eq. (7) reads as

$$\tau\begin{pmatrix} w_0 \\ w_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \end{pmatrix} \cong \begin{pmatrix} i\phi w_0 \\ i\phi w_1 \\ \vdots \end{pmatrix} \quad (9)$$

When the invariant tori survive and there is a stable frequency, we see that Eq. (9) requires

$$i\phi = \frac{w_1}{w_0} \cong \frac{w_2}{w_1} \cong \frac{w_3}{w_2}\ldots \quad (10)$$

Therefore only those vectors $\mathbf{W}_0$ which satisfy Eq. (10) with ϕ a real number represent a motion with a stable frequency given by a phase advance $\mu+\phi$ every turn. From $w_0 = u_0\mathbf{Z}_0$, we can see that ϕ is determined by the initial value $\mathbf{Z}_0$. Hence μ represents the zero amplitude tune while ϕ is the amplitude dependent tuneshift. Even though $w_0, w_1, w_2, \cdots$ all behave like action-angle variables, they have different power orders of monomials of z, z^*, and hence represent approximation of the action-angle variable to different precisions. For example, in the case of a up to 7^{th} order square matrix for a 1D system, w_0 has terms of order from 1^{st} to 7^{th} order, w_1 has terms of order from 3^{rd} to 7^{th} order while w_3 has only a very small 7^{th} order term $z(zz^*)^3$. Thus w_3 provides very little information about the rotation in the phase space while w_0 has very detailed information.

Stable motion means the invariant tori can survive with multiple turns. Applying Eq. (6) n times, we obtain

$$\mathbf{W}_n = e^{in\mu\mathbf{I}+n\tau}\mathbf{W}_0 = e^{in\mu}e^{n\tau}\mathbf{W}_0. \quad (11)$$

After some algebra computation, we recognize that a stable motion requires

$$\Im(\phi) \equiv \Im(-\frac{iw_1}{w_0}) \approx 0; \Delta \equiv \frac{w_1}{w_0} - (\frac{w_1}{w_0})^2 \approx 0. \quad (12)$$

These conditions are referred as "coherence conditions". Clearly w_0, ϕ, and Δ are all functions of initial value of z, z^*. For a given initial value of w_0, the deviation of the real part of ϕ from a constant is the tune fluctuation, while the imaginary part of ϕ gives "amplitude fluctuation", i.e., the variation of $r = |w_0|$ after many turns. The non-zero Δ indicates a deviation from "coherent state", seems to be related to the Liapunov exponents.

APPLICATION

In the following we give an example of applying the square matrix method to manipulate phase space trajectory to optimize storage ring dynamic aperture. As described before, the action defined with Courant-Snyder variable is no longer constant when nonlinearity dominates over linear dynamics. There is a significant deviation from circles in the Poincaré cross-section. We characterize the deviation as a measure of system nonlinearity. When the deviation is large, particles receive much larger nonlinear kicks, and hence the motion becomes more chaotic, or even unstable. The border of stable motion is defined as dynamic aperture. The goal of optimization is to reduce the deviation in order to ensure a sufficient dynamic aperture. The philosophy of minimization is equivalent to optimize the system so that initial coordinates sitting on the flat linear Courant-Snyder action planes after transferred by $\mathbf{M}$ can be mapped to the new approximate invariants flat planes respectively, and vice versa.

The example is to optimize the National Synchrotron Light source - II lattice with a chromaticity +7 in both horizontal and vertical planes. The NSLS-II lattice layout is described in ref. [2]. Usually a lattice with high positive linear chromaticity is preferable because it can provide additional damping to stabilize high current beam. But it is also more challenge for dynamic aperture optimization due to strong nonlinear sextupoles. First, the linear chromaticity is tuned to +7 with dispersive sextupoles, the knobs left for optimization are 6 families harmonic sextupoles located in non-dispersive sections. For this specific example, we selected 3 sets of initial values $(x_0[mm], p_x[mrad], y_0[mm], p_y[mrad]) = (25, 0, 5, 0), (10, 0, 2, 0)$ and $(3.5, 0, 3, 0)$. For each set, we cast totally 64 initial coordinates uniformly distributed on the tori with constant Courant-Snyder actions. The new actions $r_{x,y}$ after transferred by $\mathbf{M}$ for these 64 points are computed under different sextupole configurations, and they are not in flat planes any longer due to nonlinearity. A deviation defined as $\frac{\Delta r}{r} = \frac{max(r_i)-min(r_i)}{\bar{r_i}}$ is chosen as optimization objective. Here $\bar{r_i}$ is the average over all 64 r_i in a same torus. Here multi-objective genetic algorithm (MOGA) [3] was adopted to optimize for 3 sets initial values (totally 6 tori) equally and simultaneously. Actually the choice of initial values is not unique. The question about how many sets should be used, and how many points should be casted inside each set, is open for future exploration. With 4000 populations for each generation, the MOGA converges to a stable solution with a minimum $\sum(\frac{\Delta r}{r})_j$ after multiple generations, here we sum the deviations over all three sets $j = 1, 2, 3$.

Now we discuss the beam dynamics for this optimized lattice with both simulation and experimental measurement. First we compare two lattice configurations, which were optimized with the nonlinear driving terms up to the 2^{nd} order, and with the square matrix method we introduce here. Two configurations' linear optics and chromatic sextupoles settings are exactly same, only the geometric sextupoles excitations are different. In Fig. 1, we show the simulation trajectories of 5 particles starting with same initial conditions in phase space $y - p_y$ for these two settings. The left plot is the result of minimizing the driving terms, and the right one is obtained with the square matrix method. Different color represents different initial conditions. The initial amplitudes for these 5 particles gradually increase from 10 to 20mm in the horizontal plane, and from 1 to 3mm in the vertical plane, so the x-motion is nonlinearly coupled into y-motion, generating complicated motion in $y - p_y$ plane. It is obvious that even though particles with the same initial conditions can survive under both lattice settings, the optimization with the square matrix method clearly reduces the nonlinearity of the system significantly.

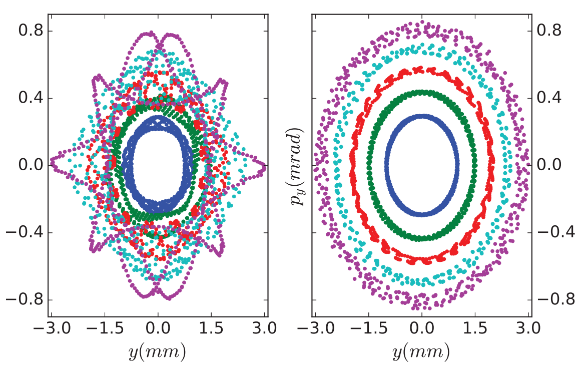

Figure 1: Comparison of simulated trajectories of $y - p_y$ in Poincaré cross-section under two sextupole configurations

It is very interesting to take a close look the tune footprint of this optimized lattice as shown in Fig. 2. We are surprised to observe a huge tune-shift-with-amplitude in both horizontal and vertical planes. And more important is that particles can survive in passing many resonances, which were regarded as the "forbidden" resonances, such as $3\nu_x = n$. Under this sextupole configuration, 100% injection efficiency has been achieved with 9mm off-axis incident beam.

Over several decades, ring designers comply with a basic "principle" – to choose a fractional tune far away from low order resonances, and then to confine tune-shift-with-amplitude to a narrow range. However, the minimization of the deviation from invariant tori produces some solutions which obviously violate this "principle". In Fig. 3, we show that the simulated horizontal phase trajectory in the Poincaré cross-section when a particle sits on a third order resonance.

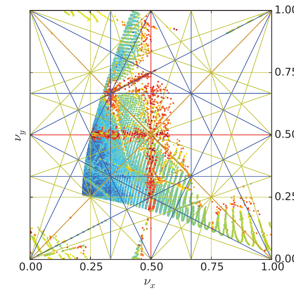

Figure 2: Frequency map analysis at x-y planes

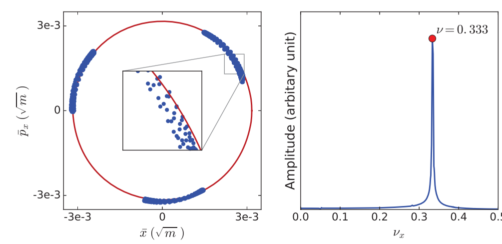

Figure 3: particle stays stably on the third order resonance

In the left plot, the red line represents a torus with a fixed Courant-Snyder action J_x, the blue dots are the simulated coordinates with the symplectic tracking code "elegant" [4]. We can see that the third order resonance has been cancelled almost perfectly. The frequency spectrum (right) of simulated data shows that the particle can stay calmly and stably on the resonance $3\nu_x = n$. Further exploration to understand beam nonlinear behavior in the vicinity of resonances is under way. Thus far, we believe that confining tune-shift-with-amplitude in order to avoid resonances is not an absolutely necessary condition if the deviations from flat planes can be well controlled. This example also suggests a new lattice design philosophy, i.e. instead of confining tune footprint, one can tune the nonlinear knobs in order to minimize the deviations at different amplitudes of $w_{x,y}$ to optimize dynamic aperture. Of course, the aperture shrinks if we take magnets imperfections into account, but simulation shows that it still crosses the third order resonance.

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