CALCULATING SPIN LIFETIME *

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Abstract

We have extended a lattice independent code to integrate the Thomas-BMT equation over 2 hours of beam time in the presence of two orthogonal Siberian snakes. In tandem to this we have recast the Thomas-BMT equation in the presences of longitudinal dynamics, into the parametric resonance formalism recently developed to understand overlapping spin resonances [1]

INTRODUCTION

One of the important factors effecting the net polarization integrated over during RHIC store is the polarization lifetime. This value has varied between a low of about 0.5% per hour to a high of 2% per hour. However we do not yet possess a good theory to explain these losses, neither has the community been able to simulate these losses. For example direct spin tracking to simulate 1 hour of beam time would take 300 million turns in RHIC. Even if we could do 100K turns in 1 hour (which is about 4 times faster than what I have observed) it would take 125 days to do this. Even if we would allocate the time we don't have the compute resources to do this for any kind of realistic distribution.

So in leu of this, we have turned to lattice independent spin tracking methods developed previously [2]. This approach involves integrating the T-BMT equation using only several spin resonances with a unitary 4th order Gaussian quadrature integrator. In this paper we present the results from an extension of this integrator to handle longitudinal dynamics.

SPIN DYNAMICS REVIEW

The dynamics of the spin vector of a charged particle with q charge in the laboratory frame is described by the Thomas-BMT equation,

$$\frac{d\vec{S}}{dt} = \frac{q}{\gamma m}\vec{S}\times\left((1+G\gamma)\vec{B}_\perp + (1+G)\vec{B}_\parallel\right), \qquad (1)$$

$\vec{S}$ is the spin vector of a particle in the rest frame, and $\vec{B}_\perp$ and $\vec{B}_\parallel$ are defined in the laboratory rest frame with respect to the particle's velocity. $G = \frac{g-2}{2}$ is the anomalous magnetic moment coefficient, and γmc^2 is the energy of the particle. Here we neglect the electric fields. Following a standard derivation (see for example [1]) the T-BMT equation can be recast in a spinor form:

$$\frac{d\Psi}{d\theta} = -\frac{i}{2}\begin{pmatrix} f_3 & -\xi \\ \xi^* & -f_3 \end{pmatrix}\Psi. \qquad (2)$$

Where $\xi(\theta) = F_1 - iF_2$ and $f_3 = (1+F_3)$ with,

$$\begin{aligned} F_1 &= -\rho z''(1+G\gamma) \\ F_2 &= (1+G\gamma)z' - \rho(1+G)\left(\frac{z}{\rho}\right)' \\ F_3 &= -(1+G\gamma) + (1+G\gamma)\rho x''. \end{aligned} \qquad (3)$$

Here, θ is the orbital angle that remains constant outside the bends. Although the spinor function Ψ is similar in form to the quantum-mechanical-state function, in this case $\vec{S}$ is a classical vector. However, as in the former case, this two-component spinor is defined,

$$\Psi = \begin{pmatrix} u \\ d \end{pmatrix}. \qquad (4)$$

u and d are complex numbers representing the up- and down-components. The components of the spin vector become

$$\begin{aligned} S_1 &= u^*d + ud^* \\ S_2 &= -i(u^*d - ud^*) \\ S_3 &= |u|^2 - |d|^2. \end{aligned} \qquad (5)$$

Because $H = (\vec{\sigma}\cdot\vec{n})$ is hermitian,

$$|\vec{S}| = |u|^2 + |d|^2 = \Psi^\dagger\Psi \qquad (6)$$

and the magnitude of the spin-vector remains constant. We chose the normalization condition for the spinor function to be $\Psi^\dagger\Psi = 1$.

When evaluating the cumulative effect of the lattice on the spin, the standard approach is to expand $F_1 - iF_2$ into a Fourier series:

$$\xi(\theta) = F_1 - iF_2 = \sum_K \varepsilon_K e^{-iK\theta} \qquad (7)$$

wherein the Fourier coefficient or resonance strength ε_K is given by the following:

$$\varepsilon_K = -\frac{1}{2\pi}\oint\Big[(1+G\gamma)(\rho z'' + iz') - i\rho(1+G)\left(\frac{z}{\rho}\right)'\Big]e^{iK\theta}d\theta. \qquad (8)$$

Here, K is the resonance spin tune. Also usually the $(1+G\gamma)\rho x''$ term is ignored to first order. Since θ is constant in a region without dipoles, it is usually clearer to express the resonance integral in terms of s:

$$\varepsilon_K = -\frac{1}{2\pi}\oint\left[(1+G\gamma)(z'' + \frac{iz'}{\rho}) - i(1+G)\left(\frac{z}{\rho}\right)'\right]e^{iK\theta(s)}ds. \qquad (9)$$

* Work supported by the US Department of Energy under contract number DE-SC0012704. This research used resources of the National Energy Research Scientific Computing Center (NERSC), which is supported by the Office of Science of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231.

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The amount of depolarization caused by acceleration through any given singular spin resonance can be evaluated using the Froissart-Stora formula [3]

$$\frac{P_f}{P_i} = 2e^{-(\pi|\varepsilon_K|^2/2\alpha)} - 1, \tag{10}$$

where,,

$$\alpha = \frac{1}{\omega_{rev}} \frac{d\nu_s}{dt} \tag{11}$$

is the spin tune crossing rate dvided by the angular revolution frequency ω_{rev}, and $\frac{P_f}{P_i}$ is the ratio of initial vertical to final vertical polarization. For a flat orbit in a constant verical field $\alpha \approx d(G\gamma)/d\theta$. The Froissart-Stora formula represents a solution to the T-BMT equation for the special case of crossing an isolated spin resonance.

LATTICE INDEPENDENT INTEGRATION

We have previously developed a code to integrate the 2D spinor form of the Thomas-BMT equation (Eq. (2)) [2] where in our case the f_3 term is set to $G\gamma$ neglecting the $(1 + G\gamma)\rho x''$ term as mentioned above.

Using a 4th order Magnus Gaussian quadrature integrator described in [4] we can integrate Eq. (2) for an arbitrary $\xi(\theta)$. In this code the effect of snakes and rotators are handled separately and are added into the lattice as thin spin kicks.

However as long as the system was held away from a low order snake resonance, integrating this system with any combination of intrinsic and imperfection resonances over 600 million turns showed no mechanism for polarization loss.

So we introduced the effect of longitudinal motion into our system. To understand effect of longitudinal motion on spin we need to first understand Its effect on transverse motion. In the simplest case a single particle's transverse motion can be characterized by

$$\frac{d^2Y(s)}{ds^2} + \omega_\beta^2/c^2 Y(s) \quad = \quad 0. \tag{12}$$

Here Y is the transverse position of the particle, s the longitudinal coordinate and ω_β gives the angular betatron frequency, which is just the angular revolution frequency ω_0 times Q the betatron tune. Solutions give transverse harmonic motion oscillating with the betatron tune Q each revolution. However if the particle resides in a rf-bucket one must consider its longitudinal motion inside of the bucket as well and equation of motions become,

$$\frac{d^2Y(s,\delta,z_L)}{ds^2} + \omega_\beta^2(\delta)/c^2 Y(s,\delta,z_L) \quad = \quad 0. \tag{13}$$

Here z_L defines the longitudinal position relative to the center of the rf-bucket, and $\delta = \Delta p/p$ the relative momentum difference from the "on momentum" particle. If we expand the betatron frequency to first order in δ we obtain.

$$\omega_\beta(\delta) = \omega_0 Q + \xi_y \omega_0 \delta. \tag{14}$$

Here $\xi_y = \frac{dQ}{d\delta}$ is called the chromaticity. We can also approximate the longitudinal motion inside the rf-bucket using,

$$\begin{aligned} \delta(s) &= \frac{-\omega_s}{\eta c} r \sin(\omega_s s/c + \phi) \\ z_L(s) &= r\cos(\omega_s s/c + \phi). \end{aligned} \tag{15}$$

Here $\omega_s = \omega_0 Q_s$ known as the synchrotron angular frequency is the angular frequency of the longitudinal motion, and ϕ is the phase of the synchrotron motion, η the phase-slip factor defined as the fractional change of the revolution period per unit of δ. We have also defined $r^2 = z_L(s)^2 + (\frac{\eta c}{\omega_s})^2 \delta(s)^2$.

Following [5], an approximate solution to Eq. (13) has the form,

$$Y(s,\delta,z_L) \approx Ae^{\pm i\Phi(s,\delta,z_L)} \tag{16}$$

where A is the constant amplitude and,

$$\begin{aligned} \Phi(s,\delta,z_L) &= \int_0^s ds'(\omega_0 Q/c + \omega_0 \xi_y \delta(s)/c) \\ \Phi(s,\delta,z_L) &= \omega_0 Qs/c + \frac{\xi_y \omega_0}{\eta c}(z_L(s) - z_L(0)) \\ \Phi(s,\delta,z_L) &= \omega_0 Qs/c + \frac{\xi_y \omega_0 z_L(0)}{\eta c}(\cos(\omega_s s/c) - 1) + \\ &\quad \frac{C_y \delta(0)}{Q_s}\sin(\omega_s s/c). \end{aligned} \tag{17}$$

Here we recast the phase in terms of initial $z_L(0)$ and $\delta(0)$ using the transformation from the frame rotating with the synchrotron frequency,

$$\begin{pmatrix} z_L(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} \cos(\omega_s s/c) & \frac{\eta c}{\omega_s}\sin(\omega_s s/c) \\ -\frac{\omega_s}{\eta c}\sin(\omega_s s/c) & \cos(\omega_s s/c) \end{pmatrix} \begin{pmatrix} z_L(0) \\ \delta(0) \end{pmatrix}. \tag{18}$$

We can now model the effect of longitudinal motion via a phase modulation applied to the betatron phase term which appears in $\xi(s)$ as,

$$\sum_K \epsilon_K e^{-iK\theta + i\frac{\xi_y \omega_0 \tau_0}{\eta}(1-\cos(\omega_s s/c))}. \tag{19}$$

Here we have set $z_L(0) = \tau_0 c$. We choose the initial $\delta(0) = 0$ since that will only alter the constant initial phase and shouldn't contribute to the dynamics which drive polarization lifetime.

In addition to the phase effect there is also a direct energy modulating effect which should modify our value for $G\gamma$ as follows,

$$G\gamma(s) = (G\gamma_0 + \alpha\theta)(1 + \delta(\theta)); \tag{20}$$

Since we have chosen $\delta(0) = 0$ this makes our $\delta(s) = \omega_s \tau_0 \sin(\omega_s s/c)/\eta$.

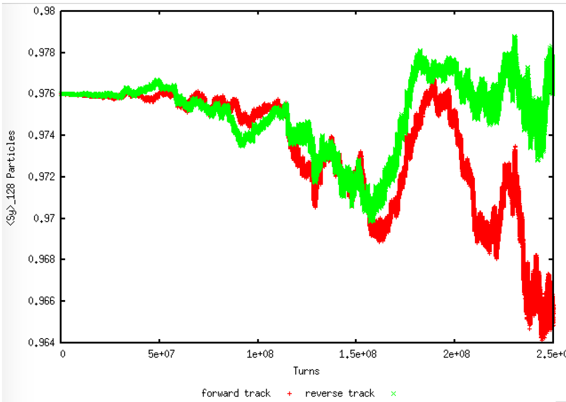


Figure 1: Average evolution of polarization for forward and reverse tracking for 1 hour (1 hour forward, 1 hour reverse). Path's are diverged initially because reverse tracking staggered by one turn.

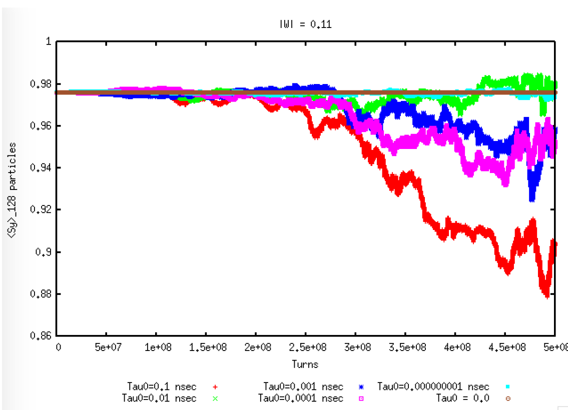


Figure 2: turn-by-turn average evolution of polarization for different initial τ_0

NUMERICAL ERROR CHECK

Since we are running our integrator over 10^8 turns one important concern is the development of roundoff errors and deviations from unitarity. Checks of unitarity found deviations out to two hours of less than 1×10^{-8}. We also performed reverse tracking. Here we took 128 particles out to 1 hour (2.5^8 turns) and then reverse tracked to recover our starting spin values to within 5×10^{-6} (see Fig. 1)

DRIVERS OF POLARIZATION LIFETIME

As can be seen in Fig. 2 the integration of the T-BMT equation for a single resonance with two orthogonal snakes, over 2 hours of beam time show no observable polarization loss. The introduction of longitudinal motion however changes this situation very dramatically. Additionally the threshold appears to be less than $10^{-16}sec$ in τ_0 amplitude. The role of the update from Eq. (20) appears negligible in relation to the polarization lifetime. This is because omitting or including the update in the simulation didn't seem to change or cause the onset of a polarization decay.

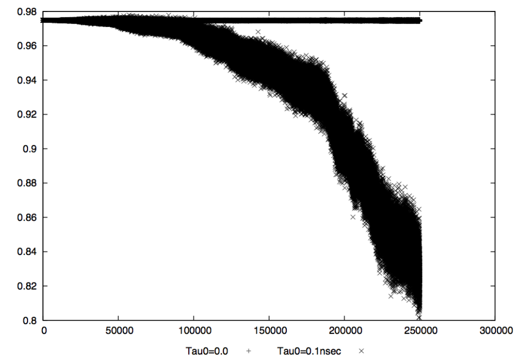


Figure 3: turn-by-turn average evolution of polarization for different initial τ_0 with four neighboring resonances.

If we now consider the evolution of the average spin vector in the presence of up to four additional nearby intrinsic resonances we see that like the single resonance case, it is only with the introduction of longitudinal motion that a discernable polarization lifetime appears (see Fig. 3)

If we also consider the response of the lifetime to different factors related to longitudinal motion. We see that it is rather insensitive. So for example we saw that with only $10^{-16}sec$ timing offset from the bucket center polarization decay sets in. However above this changes of initial τ_0, synchrotron tune and chromaticity show little correlation with increased decay rates (see Fig. 4).

While longitudinal dynamics seems essential to trigger the process of polarization decay,the response to changes in longitudinal parameters seems rather insensitive. In the range of parameters which is reasonable for synchrotron tune, momentum offset and chromaticity decay rates don't seem to change significantly.

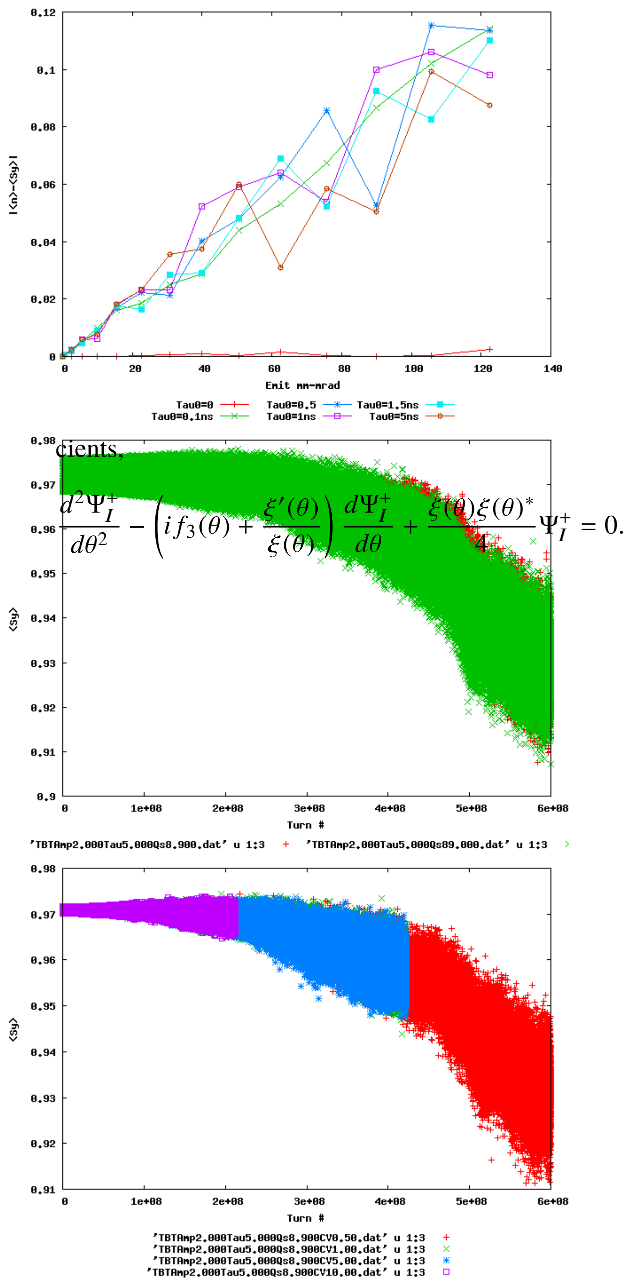
However as you might have noticed in the top figure in Fig. 4 polarization lifetime is very sensitive to the magnitude of the nearby intrinsic spin resonances.

We also can see in Fig. 5 that beyond a threshold of 0.01 resonance strength, the imperfection resonance strength for both nearby imperfections can play an important role in driving polarization decay rates.

Using the curves similar to those shown in Fig. 4 and 5 one can provide an estimate for the lifetime by interpolating loss versus emittance and then integrating this over the emittance distribution,

$$\begin{aligned} \rho(\epsilon) &= \frac{e^{-\frac{\epsilon}{2\epsilon_0}}}{2\epsilon_0} \\ Pol_{loss} &= \int_0^{\epsilon_{max}} d\epsilon \rho(\epsilon) f(\epsilon). \end{aligned} \tag{21}$$

Here $f(\epsilon)$ is a fit the polarization loss versus emittance curves. We found for the FY15 100 GeV lattice, lifetime estimates of between 0.14 to 0.4% per hour for an ϵ_0 between 10 to 30 $\pi mm - mrad$ rms Normalized.



cients,

$$\frac{d^2\Psi_I^+}{d\theta^2} - \left(i f_3(\theta) + \frac{\xi'(\theta)}{\xi(\theta)}\right)\frac{d\Psi_I^+}{d\theta} + \frac{\xi(\theta)\xi(\theta)^*}{4}\Psi_I^+ = 0. \quad (25)$$

Figure 4: Polarization deviation from stable spin direction after 2 hours for different initial τ_0 with four neighboring resonances (top). Turn-by-turn average evolution of polarization for different initial synchrotron tune $qs = 8.9\times10^{-4}$ and 89.0×10^{-4} (middle). Turn-by-turn evolution for chromaticity = 0.5,1,5 and 10. These are all for stored $G\gamma = 191.5$ and vertical tune of 30.693 with four intrinsic resonances calculated from the FY15 100 GeV pp lattice.

CONJECTURE OF MECHANISM FOR POLARIZATION LOSSES

If we consider only a single spin resonances with synchrotron motion then Eq. (24) becomes,

$$\xi(s) \quad = \quad a_1 e^{-iK_1\theta - i\phi_1 + i\frac{g_0}{q_s}(\cos(q_s\theta)-1)}, \qquad (22)$$

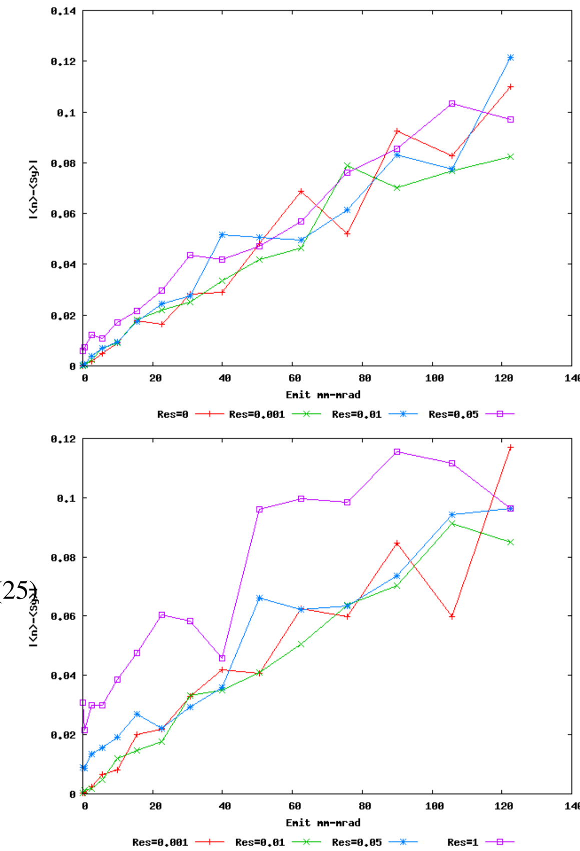


Figure 5: Polarization deviation from stable spin direction after 2 hours for different initial imperfection resonances for the $G\gamma = 191$ (top) and for $G\gamma = 192$. These are all for stored $G\gamma = 191.5$ and vertical tune of 30.693 with four intrinsic resonances calculated from the FY15 100 GeV pp lattice.

where we use $\epsilon_1 = a_1 e^{-i\phi_1}$ and $g_0 = \frac{\xi\omega_0\tau_0 q_s}{\eta}$. Starting from Eq. (2) we can move to the Interaction frame using the transformation:

$$\begin{aligned} \Psi(\theta) &= e^{-\frac{i}{2}\int_0^\theta f_3(t)dt\hat{\sigma}_z}\Psi_I(\theta) \\ \hat{\xi}(\theta) &= \xi(\theta)e^{i\int_0^\theta f_3(t)dt}, \end{aligned} \qquad (23)$$

This yields the following:

$$\frac{d\Psi_I^+}{d\theta} = \frac{i}{2}\hat{\xi}\Psi_I^- \quad \frac{d\Psi_I^-}{d\theta} = \frac{i}{2}\hat{\xi}^*\Psi_I^+. \qquad (24)$$

These equations can be cast into a standard 2^{nd} order homogeneous linear differential equation with variable coefficients,

$$\frac{d^2\Psi_I^+}{d\theta^2} - \left(i f_3(\theta) + \frac{\xi'(\theta)}{\xi(\theta)}\right)\frac{d\Psi_I^+}{d\theta} + \frac{\xi(\theta)\xi(\theta)^*}{4}\Psi_I^+ = 0. \qquad (25)$$

where we introduce $\Delta = K_0 - K_1$ and also show the equation for both $\Psi_I^\pm$. Following the approach used in [1] we can recast this equation as,

$$y^{\pm\prime\prime} + \left(\left(\frac{a_1^2}{4} + \frac{\Delta^2}{4}\right) - \frac{g_0}{2}\left(\mp i q_s \cos(q_s\theta) + \Delta \sin(q_s\theta)\right) + \frac{g_0^2}{4}\sin^2(q_s\theta)\right) y^{\pm} = 0 \quad (27)$$

In this case have transformed Eq. (26) using the following standard change of variables,

$$\begin{aligned} \beta^{\pm}(\theta) &= \pm i g_0 \sin(q_s\theta) \mp i\Delta \\ \pm i D(\theta) &= \frac{1}{2}\int^{\theta} dx \beta^{\pm}(x) \\ y^{\pm}(\theta) &= \Psi_I^{\pm}(\theta) e^{i\pm D(\theta)} \end{aligned} \quad (28)$$

In the form of Eq. (27) we can see clearly that we have now an oscillating kernel and thus an equation which has an infinite number of parametric resonances populating the parameter space where $a_1^2/4 + \Delta^2/4 = (nqs)^2/4$ and $n = 1, 2, 3...$ Since we are considering polarization lifetime issues over very large number of turns, even the very high order parametric resonances can potentially drive losses now where as before the introduction of the synchrotron motion there was no oscillating kernel and thus no mechanism for polarization loss.

While this logic seems correct for the single resonance model which we have outlined above. It is still unclear why in the case of multiple overlapping spin resonances in the absence of synchrotron motion this is not the case? We think this might be due to cancelations of the parametric driving resonances terms which could occur in the presence of two orthogonal snakes. The above analysis of course didn't include the effect of snakes. Since the intrinsic resonances are a property of the lattice, snakes are well suited to cancel their effects. However synchrotron motion is a property of the RF system and thus it lacks the periodicity with respect to the lattice which the intrinsic resonances do.

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