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ADJOINT APPROACH TO ACCELLERATOR LATTICE DESIGN*

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Abstract

Traditionally, accelerator lattices are designed using computer codes that solve the equations of motion for charged particles in both prescribed and self-consistent fields [1, 2]. These codes are run in a mode in which particles enter a lattice region and travel through the lattice for a finite distance. Various figures of merit (FoMs) are evaluated, and the lattice is then optimized by varying the positions and strengths of the focusing elements. [3, 4] This optimization is done in a high dimensional parameter space, requiring multiple simulations of the particle trajectories. We propose to alter the design process using "adjoint" techniques [5]. Incorporation of an "adjoint" calculation of the trajectories and self-fields can, in several runs, determine the gradient in parameter space of a given FoM with respect to all lattice parameters. It includes naturally self-fields and can be embedded in existing codes. The theoretical basis for the method and an application will be presented.

LATTICE OPTIMIZATION

Design and optimization of accelerator lattices is carried out using sophisticated numerical models. [3, 4] The current state of the art codes follow a large number of simulation particles through the combined magnetic fields of the focusing magnets and the electric and magnetic self-fields of the charged simulation particles [1,2]. Typically, particles are followed for a finite distance, their ending coordinates are recorded, and a figure of merit (FoM) quantifying the quality of confinement is assessed. To optimize the FoM the strengths and locations of the focussing magnets are varied and the gradient of the FoM in parameter space is determined. The parameters are then adjusted to increase the FoM and the process is repeated. As there are many parameters describing the lattice, direct determination of the gradient of the FoM is computationally expensive. We present here an alternative (adjoint) approach that reduces dramatically the number of simulations needed to determine the gradient.

ADJOINT APPROACH

The adjoint approach is based on a form of reciprocity implicit in Hamilton's equations and is associated with the symmetry of the governing equations under time reversal [5]. We describe briefly a paraxial, model system here in which charged particles move in a four-dimensional phase space $(\mathbf{x}_\perp,\mathbf{p}_\perp)$ in time-independent fields with distance along the path acting as time. After a canonical transformation, the motion is governed by Hamilton's equations with axial momentum serving as the Hamiltonian,

$$\frac{d\mathbf{x}_\perp}{dz}=-\frac{\partial P_z}{\partial \mathbf{p}_\perp},\quad \frac{d\mathbf{p}_\perp}{dz}=\frac{\partial P_z}{\partial \mathbf{x}_\perp}, \tag{1}$$

where

$$P_z(\mathbf{x}_\perp,\mathbf{p}_\perp,z)=P_{z0}-q\Phi_{eff}(\mathbf{x}_\perp,z)/v_{z0}-\frac{1}{2}\left|\mathbf{p}_\perp\right|^2/P_{z0}. \tag{2}$$

Here $\Phi_{eff}(\mathbf{x}_\perp,z)=\Phi-v_{z0}A_z/c$ includes both electrostatic and magnetic contributions and satisfies the Poisson equation $-\nabla_\perp^2\Phi_{eff}=4\pi\gamma_{z0}^{-2}\rho$, where $\rho(\mathbf{x}_\perp,z)$ is the beam charge density. We assume for simplicity that all particles have the same axial velocity. The portion of the potential due to the focussing magnets is imposed through A_z as a boundary condition on the solution for the effective potential.

Let us assume that we have solved the self - consistent nonlinear equations for beam propagation over a distance L. This is our reference solution for which we evaluate the FoM. We now consider two perturbations of this solution. One perturbation, which we label with a superscript 1, is the true solution to the case in which the effective potential at the wall is changed by a small amount reflecting a small change in the focusing magnets. The other perturbation, which we label with superscript 2, is the adjoint solution. The change in symplectic area entering and leaving the region $0<z<L$, for these two perturbations is then expressed (see Ref. 5),

$$\begin{aligned}&\sum_j I_j\left(\delta\mathbf{p}_{j\perp}^{(1)}\cdot\delta\mathbf{x}_{j\perp}^{(2)}-\delta\mathbf{p}_{j\perp}^{(2)}\cdot\delta\mathbf{x}_{j\perp}^{(1)}\right)_0^L\\&=\frac{q\gamma_{z0}^2}{4\pi}\int dl\,dz\left[\left(\mathbf{n}\cdot\nabla_\perp\delta\phi_{eff}^{(1)}\right)\delta\phi_{eff}^{(2)}-\left(1\leftrightarrow 2\right)\right]\end{aligned}. \tag{3}$$

Here I_j is the current associated with particle j, and the integral on the right is carried out over the transverse boundary where the effective potential is specified. Relation (3) can be regarded as an extension of Green's theorem to include the effect of dynamic charge.

The approach now is to pick conditions on the adjoint solution, with superscript 2, such that Eq. (3) becomes an evaluation of the change in the figure of merit associated with the true solution, the one with superscript 1. As an example, consider a figure of merit based on the coordinates of the particles at the exit plane, $z = L$, $F=\sum_j I_j f(\mathbf{x}_{\perp j},\mathbf{p}_{\perp j})\big|_{z=L}$ The change in this quantity for the true solution is given by

$$\delta F=\sum_j I_j\left[\delta\mathbf{x}_{\perp j}^{(1)}\cdot\partial f/\partial\mathbf{x}_\perp+\delta\mathbf{p}_{\perp j}^{(1)}\cdot\partial f/\partial\mathbf{p}_\perp\right]_{z=L}. \tag{4}$$

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If we assume there is no perturbation of the true orbit at z=0, and we select the final coordinates of the adjoint solution to be given by

$$\delta\mathbf{p}_{\perp j}^{(2)} = -\varepsilon_1 \partial f / \partial \mathbf{x}_{\perp j}, \quad \delta\mathbf{x}_{\perp j}^{(2)} = -\varepsilon_1 \partial f / \partial \mathbf{p}_{\perp j},$$

then the left-hand side of (3) becomes proportional to the change in the FoM. Here ε_1 is a small constant selected to insure that the adjoint solution is a liner perturbation. If we then take the adjoint solution to satisfy a Dirichelet condition, $\delta\Phi_{eff} = 0$ on the radial boundary, Eq. (3) becomes

$$\delta F = -\frac{q\gamma_{z0}^2}{4\pi\varepsilon_1}\int dl\, dz\left[\left(\mathbf{n}\cdot\nabla_\perp \delta\phi_{eff}^{(2)}\right)\delta\phi_{eff}^{(1)}\right]. \tag{5}$$

Thus, the change in the FoM for an arbitrary change in the focussing magnets, as controlled by $\delta\Phi_{eff}^{(1)}$, is found by projecting onto the normal derivative of the adjoint potential. The adjoint problem only needs to be simulated once to determine how the FoM will change for any arbitrary change in the focusing magnets. It does require slightly perturbing the particle coordinates at the exit and running the simulation backward in z [5].

Model Problem

The above procedure has been applied in a particle code to determine the sensitivity of beam properties to electrode shape [5]. Here we apply the method to the sensitivity of beam properties in an accelerator lattice in a simpler system, namely the envelope equations [6]. We consider the evolution of the transverse RMS widths *a, b* and associated derivatives $p_a = da/dz$, $p_b = db/dz$. These satisfy Hamilton's equations

$$\frac{d}{dz}p_{(a,b)}(z) = -\frac{\partial H}{\partial(a,b)}, \quad \frac{d}{dz}(a,b) = \frac{\partial H}{\partial(p_a,p_b)}, \tag{6}$$

where

$$H = \frac{p_a^2+p_b^2}{2} - \frac{1}{2}\left(k_x a^2 + k_y b^2\right) - 2K\ln(a+b) + \frac{\varepsilon^2}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right),$$

is the Hamiltoniain, $k_x(z)=-k_y(z)$ are profiles of quadrupole field strength, K is a constant space charge parameter (proportional to the beam's line-charge density) and ε is the admittance. Equation (6) generates the following differential equation for the RMS width $a(z)$,

$$\frac{d^2a(z)}{dz^2} = k_x a + \frac{2K}{a+b} + \frac{\varepsilon^2}{a^3},$$

and a similar equation for $b(z)$.

We then imagine, as before, two perturbations to this system, a true perturbation with superscript 1 and an adjoint perturbation with superscript 2. Due to the Hamiltonian property of the governing system, these lead to a relation similar to (3)

$$\begin{aligned}&\left\{a^{(2)}p_a^{(1)} + b^{(2)}p_b^{(1)} - a^{(1)}p_a^{(2)} - b^{(1)}p_b^{(2)}\right\}_0^L \\ &= \int_0^L dz\left(k_x^{(1)}aa^{(2)} + k_y^{(1)}bb^{(2)} - (1\leftrightarrow 2)\right)\end{aligned} \tag{7}$$

We have dropped the notational δ signifying a small perturbation. Equation (7) can be manipulated into the same form as Eq. (5) for evaluations of FoMs where the true perturbations vanish at z=0. Alternatively, if one considers solutions to the true problem that are periodic in z with period L, then the left hand side of Eq. (7) takes the form

$$\begin{aligned}&\left\{\Delta a^{(2)}p_a^{(1)} + \Delta b^{(2)}p_b^{(1)} - a^{(1)}\Delta p_a^{(2)} - b^{(1)}\Delta p_b^{(2)}\right\} \\ &= \int_0^L dz\left(k_x^{(1)}aa^{(2)} + k_y^{(1)}bb^{(2)} - (1\leftrightarrow 2)\right)\end{aligned}, \tag{8}$$

where the true variables are measured at $z = 0$ or L, and the notation $\Delta a^{(2)} = a^{(2)}(L) - a^{(2)}(0)$, applies to all the adjoint variables. Again, Eq. (8) can be used to evaluate the change in FoMs of the form $F(a,b,p_a,p_b)\big|_{z=0}$, provided the appropriate values $\Delta a^{(2)}$, etc. are found. This can be done using superpositions of 4 independent adjoint solutions.

Equation (8) also presents the possibility of determining the gradient of integrated FoMs of the form $F = \int_0^L dz\, f(a(z),b(z),z)$. Here the change in F is given by

$$\delta F = \int_0^L dz\left(a^{(1)}\frac{\partial f}{\partial a} + b^{(1)}\frac{\partial f}{\partial b}\right). \tag{9}$$

If we consider periodic adjoint solutions with specified added focusing,

$$k_x^{(2)} = \frac{\partial f}{a\partial a}, \quad k_y^{(2)} = \frac{\partial f}{b\partial b},$$

then the left side of (8) vanishes, but the added focusing appears on the right-hand side giving

$$\delta F = \int_0^L dz\left[\left(k_x^{(1)}aa^{(2)} + k_y^{(1)}bb^{(2)}\right)\right]. \tag{10}$$

Example

We consider a simple example where the lattice consists essentially of two quadrupole magnets of opposite sign equally spaced over a distance of z = 0.08 - 0.40 m. The profile of $k_x(z) = -k_y(z)$ is shown as a black line in Fig. 1, showing that the period starts and ends in the center of one of the magnets. We chose for this example $K = 10^{-5}$

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and $\varepsilon = 7.6\times10^{-6}$ m. We adopt for illustration as a figure of merit

$$F = \frac{1}{2}\left[p_a^2 + p_b^2 + \lambda^{-2}\left(a^2 + b^2\right)\right]_{z=L}, \qquad (11)$$

where λ = 10 m in this example. If F is evaluated where both momenta vanish, it corresponds roughly to the cross sectional area of the beam. We then consider matched solutions of the envelope equations, and ask what is the change in F for an arbitrary change in the profile of the quadrupole magnetic field. The change in F is given by,

$$\delta F = p_a p_a^{(1)} + p_b p_b^{(1)} + \lambda^{-2}\left(a a^{(1)} + b b^{(1)}\right)$$

By inspecting Eq. (8) we see that if we find non-matched, perturbed solutions under the same focusing fields, $\left(k_x^{(2)} = -k_y^{(2)} = 0\right)$, and that have as boundary conditions $\Delta a^{(2)} = \varepsilon_2 p_a(L)$, $\Delta b^{(2)} = \varepsilon_2 p_b(L)$, $\Delta p_a^{(2)} = -\varepsilon_2 \lambda^{-2} a(L)$, and $\Delta p_b^{(2)} = -\varepsilon_2 \lambda^{-2} b(L)$. Then Eq. (8) reduces to

$$\delta F = \frac{1}{\varepsilon_2}\int_0^L dz\left[\left(k_x^{(1)} a a^{(2)} + k_y^{(1)} b b^{(2)}\right)\right]. \qquad (12)$$

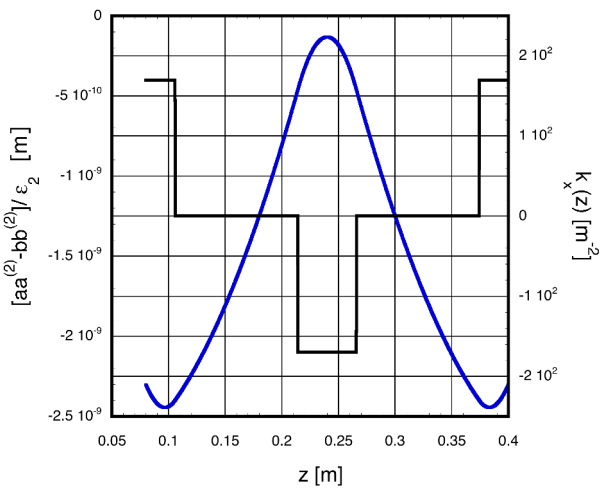

Figure 1: Plots of the profile of the quadrupole focusing force (black line) and the sensitivity of the figure of merit (11) (blue line) for the example described in the text.

Here the factor ε_2 cancels in the numerator and denominator on the right side of (12) and is introduced to signify that the variables with superscript 2 are indeed small perturbations. Finding these perturbed solutions involves four independent solves of the perturbed envelope equations and forming a linear superposition of these satisfying the boundary conditions. Once this is done (12) allows for a determination of the change in the figure of merit for an arbitrary change in the profile and strength of the focusing magnets.

We have carried out the above procedure for the example under consideration and the result, in the form of the profile of $\left[a(z)a^{(2)}(z) - b(z)b^{(2)}(z)\right]/\varepsilon_2$ appears in Fig. 1. The validity of this approach has been verified via comparison with direct calculation of the change in the figure of merit when the strength of the central quadrupole is increased by a small amount (0.1%; a value selected to be in the range of linear variations). The agreement between the two methods for calculating the change in F was within 0.2%.

CONCLUSION

Adjoint methods have the potential to greatly speed up calculations of the sensitivity of various Figures of Merit to changes in the profiles describing the focussing lattice in accelerators. Such methods can be applied to both moment and particle descriptions of the accelerated beam.

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