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A FAST METHOD TO EVALUATE TRANSVERSE COUPLED-BUNCH STABILITY AT NON-ZERO CHROMATICITY*

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Abstract

We present a dispersion relation that gives the complex growth rate for coupled-bunch instabilities at arbitrary chromaticity in terms of its value at zero chromaticity. We compare predictions of the theory to elegant tracking simulations, and show that there are two distinct regimes to stability depending upon whether the zero chromaticity growth rate is smaller or larger than the chromatic tune shift over the bunch. We derive an approximate expression that is easily solved numerically, and furthermore indicate how the formalism can be extended to describe arbitrary longitudinal potentials.

INTRODUCTION

The standard theory of coupled-bunch instabilities expands the distribution function in orthogonal synchrotron modes [1–3], which are typically then simplified by assuming that the modes do not couple. However, assuming that the modes are independent becomes poor at high chromaticity. We present a dispersion relation that is valid at arbitrary chromaticity and wakefield strength that is relatively easy to solve.

OUTLINE OF THE DERIVATION

Our theory begins with the single particle dynamics including the (linear) transverse betatron motion, the longitudinal focusing, and the chromatic coupling between the two. Our first step is to choose a new set of coordinates that approximately eliminates the chromatic coupling; this coordinate change involves the well-known "head-tail" (or chromatic) phase [4, 5]

$$\frac{\omega_0 \xi}{\alpha_c c} z = \frac{2\pi\xi}{\alpha_c c T_0} z \equiv k_\xi z, \tag{1}$$

where $\omega_0 = 2\pi/T_0$ is the revolution frequency, ξ is the chromaticity, and α_c is the momentum compaction. The head-tail phase arises because the betatron frequency depends linearly on the energy for $\xi \neq 0$, which in turn leads to the betatron phase accumulating a shift that is proportional to the longitudinal coordinate z as it performs synchrotron oscillations. Hence, the quantity $k_\xi \sigma_z$ encapsulates the chromatic tune-shift across a bunch of length σ_z.

Next, we add the long-range transverse wakefield, whose effect we will describe using the distribution function of bunch j $F_j(\mathscr{Z}; s)$, where the phase-space coordinates $\mathscr{Z} = (z, p_z, \Psi, \mathscr{J})$, and $\int d\mathscr{Z}\; F_j = 1$ for N_b bunches (i.e., $0 \le j \le N_b - 1$). The dipolar wakefield gives a kick to trailing particles that is proportional to the displacement of the leading particle, and the total kick is obtained by summing the wakefields over all bunches in the ring and over all previous turns. Defining the equilibrium centroid spacing between bunch n and j to be $L_{n,j}$ with $L_{n,j} > 0$ if $j > n$ and $L_{n,j} = -L_{n,j}$ if $j \le n$, the potential due to the dipole wakefield is

$$\mathscr{V}_D = y \sum_{j=0}^{N_b-1} \frac{e^2 N_j}{\gamma m c T_0} \sum_{\ell=0}^{\infty} \int d\mathscr{Z}'\, y' F_j(\mathscr{Z}'; s - \ell c T_0) \times W_D[z - (z' + \ell c T_0 + L_{n,j})].$$

We expand the long-range wakefield W_D assuming it varies slowly over the bunch,

$$\begin{aligned} &\approx \sqrt{\mathscr{J}} \cos(\bar{\Psi} - k_\xi z) \sum_{j=0}^{N_b-1} \frac{2e^2 N_j}{\gamma m c T_0} \\ &\quad \times \sum_{\ell=0}^{\infty} W_D^\beta(-\ell c T_0 - L_{n,j}) \qquad (2)\\ &\quad \times \int d\mathscr{Z}' \sqrt{\mathscr{J}'} \cos(\bar{\Psi}' - k_\xi z') F_j(\mathscr{Z}'; s - \ell c T_0) \\ &= \sqrt{\mathscr{J}} \cos(\bar{\Psi} - k_\xi z) \sum_{j=0}^{N_b-1} \mathscr{W}_{n,j}(s), \qquad (3) \end{aligned}$$

where $\mathscr{W}_{n,j}$ is proportional to the kick that particles in bunch n receive due to the centroid displacement of bunch j.

In terms of the transverse and longitudinal actions $(\mathscr{J}, \mathscr{T})$ and their respective angles (Ψ, Φ), the single particle Hamiltonian is

$$\begin{aligned} \mathscr{H} &= \mathscr{H}_z(\mathscr{T}) + \frac{\omega_\beta}{c} \mathscr{J} \\ &\quad + \sqrt{\mathscr{J}} \cos[\bar{\Psi} - k_\xi z(\Phi, \mathscr{T})] \sum_{j=0}^{N_b-1} \mathscr{W}_{n,j}(s), \end{aligned} \tag{4}$$

where, the dependence on $\Psi - k_\xi z$ comes from the coordinate change using Eq. (1), $\mathscr{W}_{n,j}$ is the dipolar kick defined by (2)-(3), and $\mathscr{H}_z$ and $z(\Phi, \mathscr{T})$ depends on the rf potential.

The Hamiltonian (4) specifies the particle equations of motion within our model, and to determine multi-bunch collective stability we will consider the coupled set of Vlasov equations associated with $\mathscr{H}$. Each bunch satisfies its own Vlasov equation, and for bunch n we have

$$\begin{aligned} 0 &= \frac{\partial F_n}{\partial s} + \frac{\omega(\mathscr{T})}{c} \frac{\partial F_n}{\partial \Phi} + \frac{\omega_\beta}{c} \frac{\partial F_n}{\partial \Psi} \\ &\quad + \sum_{j=0}^{N_b-1} \mathscr{W}_{n,j} \frac{\cos(\Psi - k_\xi z)}{2\sqrt{\mathscr{J}}} \frac{\partial F_n}{\partial \Psi} \qquad (5)\\ &\quad + \sum_{j=0}^{N_b-1} \mathscr{W}_{n,j} \sqrt{\mathscr{J}} \sin(\Psi - k_\xi z) \frac{\partial F_n}{\partial \mathscr{J}}, \end{aligned}$$

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where we have introduced the amplitude-dependent longitudinal (i.e., synchrotron) frequency $\omega(\mathcal{J}) = \partial\mathcal{H}_z/\partial\mathcal{J}$. The theory is Hamiltonian, and any dissipative effects (e.g., synchrotron emission) can only be approximately included later.

Our next step is to reduce the $N_b - 1$ Vlasov equations into a coupled set of linear, ordinary differential equations for the transverse centroid positions. This process is basically done in two steps. The first step involves linearizing the problem about the equilibrium distribution function, multiplying by the transverse complex dipole displacement, and then integrating over the transverse degree of freedom. After dropping fast oscillating terms and then taking the Fourier transform, we are left with a coupled set of linear equations for the dipole-weighted longitudinal distribution function. The second step involves solving for the dipole-weighted longitudinal distribution function using the methods in [6,7], and then integrating over longitudinal phase space to obtain a set of coupled equations for the transverse dipole moments in terms of the long-range wakefield and equilibrium quantities. In terms of the (chromatic-shifted) transverse dipole moment

$$\mathcal{D}_n = e^{-i\Omega s/c} e^{-i\omega_\beta s/c} \int d\mathcal{Z}\ \sqrt{\mathcal{J}} e^{-i\Psi} e^{ik_\xi z} \tilde{f}_n(\mathcal{Z}), \qquad (6)$$

this procedure results in the following equation

$$\mathcal{D}_n = \mathcal{K}(\Omega, k_\xi) \sum_{j=0}^{N_b-1} \mathsf{M}_{n,j} \mathcal{D}_j \qquad (7)$$

where $\mathcal{K}(\Omega, k_\xi)$ depends upon the equilibrium properties and will be specified shortly, while the coupling matrix M has components

$$\mathsf{M}_{n,j} = \frac{e^2 N_{e,j}}{2\gamma mc^2 T_0} \sum_{\ell=0}^{\infty} W_D(-\ell c T_0 - L_{nj}) e^{i\ell\omega_\beta T_0}. \qquad (8)$$

The matrix M is essentially the same as the theory of Thompson and Ruth [8], and the formula (7) is particularly convenient because it divides into two pieces: one piece that is the usual matrix theory, and the other which involves the longitudinal distribution function and the chromaticity. The former can be diagonalized in the usual way, and we will assume that a matrix U has been found such that $\mathsf{U}\mathsf{M}\mathsf{U}^{-1} = \lambda\mathsf{I}$ with I the identity. Then, stability is governed by $1 = \lambda\mathcal{K}(\Omega, k_\xi)$, and we find that the two basic parameters of the theory are the $\xi = 0$ growth rate λ (encapsulating the strength of the wakefield), and the head tail frequency k_ξ (summarizing chromatic effects).

Now, we just need the expression for $\mathcal{K}$. We will do this in terms of dimensionless quantities, and therefore define

$$\hat{\lambda} = \frac{\lambda}{\alpha_c\sigma_\delta/\sigma_t}, \qquad \hat{\Omega} = \frac{\Omega}{\alpha_c\sigma_\delta/\sigma_t}, \qquad (9)$$

where σ_t and σ_δ are the rms bunch duration and energy spread, respectively, and $\alpha_c\sigma_\delta/\sigma_t$ is of order the characteristic synchrotron frequency. If we additionally assume that we can approximate the longitudinal position by its lowest order harmonic, $z(\Phi, \mathcal{J}) \approx \zeta(\mathcal{J})\cos\Phi$, then we find that $1 = \lambda\mathcal{K}(\Omega, k_\xi)$ can be written

$$\begin{aligned} 1 = -i\hat{\lambda} \int_0^\infty d\mathcal{J}\ &\frac{2\pi\bar{f}(\mathcal{J})}{\hat{\omega}(\mathcal{J})[1 - e^{2\pi i\hat{\Omega}/\hat{\omega}(\mathcal{J})}]} \\ &\times \int_0^{2\pi} d\theta\ J_0[2k_\xi\zeta(\mathcal{J})\sin(\theta/2)] e^{i\hat{\Omega}\theta/\hat{\omega}(\mathcal{J})}, \end{aligned} \qquad (10)$$

where the scaled frequency $\hat{\omega} = \omega(\mathcal{J})/(\alpha_c\sigma_\delta/\sigma_t)$ and equilibrium $\bar{f}(\mathcal{J})$. We will assume that the equilibrium is an exponential function of the energy, $\bar{f} \propto e^{-\mathcal{H}/\alpha_c\sigma_\delta^2}$.

HARMONIC RF POTENTIAL

In this section we will assume that the longitudinal potential is given by the harmonic approximation of a single rf system, for which the Hamiltonian $\mathcal{H}_z = \omega_s\mathcal{J}/c$ with the synchrotron frequency $\omega_s = \alpha_c\sigma_\delta/\sigma_t$. In this case we also have

$$\bar{f}(\mathcal{J}) = \frac{e^{-\mathcal{J}/\sigma_\delta\sigma_z}}{2\pi\sigma_\delta\sigma_z}, \quad \hat{\omega} = 1, \quad \zeta(\mathcal{J}) = \sigma_z\sqrt{\frac{2\mathcal{J}}{\sigma_\delta\sigma_z}}. \qquad (11)$$

Then, we can integrate over action in (10) to get

$$1 = \frac{-i\hat{\lambda}}{1 - e^{2\pi i\hat{\Omega}}} \int_0^{2\pi} d\theta\ e^{-k_\xi^2\sigma_z^2(1-\cos\theta)} e^{i\hat{\Omega}\theta}. \qquad (12)$$

This is related to the single-bunch result derived in [7], but is significantly simpler due to our Taylor series expansion of the long-range wakefield. Coupled-bunch stability can be found for a given long-range wakefield eigenvalue λ and chromaticity $k_\xi = \xi\omega_0/\alpha_c c$ by solving (12) for Ω. In the limit that the chromaticity vanishes we have $k_\xi = 0$ and the integration can be easily done to find that $\Omega = \lambda$. This result actually follows from the general Eq. (10), so that in the zero chromaticity limit stability is independent of the longitudinal potential, assuming that the long-range wakefield varies slowly over the bunch length. This is to be expected, since when $\xi = 0$ the transverse motion is uncoupled from the synchrotron motion. On the other hand, at non-zero chromaticity the transverse betatron frequency depends upon the particle energy and therefore on its longitudinal position in the bunch, so that the collective oscillation is more complicated when $\xi \neq 0$.

When the instability is weak, $\Im(\hat{\Omega}) \ll \omega_s$, we can derive the following approximate form

$$\hat{\Omega} \approx \frac{\hat{\lambda}}{\sqrt{2\pi}k_\xi\sigma_z}\left[1 + \frac{m}{k_\xi^2\sigma_z^2}\left(\hat{\lambda} - \frac{m}{2}\right)\right] \qquad (13)$$

for integer m. Equation (13) predicts that the coupled-bunch growth rate is reduced from its $\xi = 0$ value by an amount proportional to the chromatic phase over the bunch, and for small $\hat{\lambda}$ the reduction is by a factor $\sqrt{2\pi}k_\xi\sigma_z$. Additionally, the instability depends upon m: for sufficiently small coupled-bunch eigenvalue $\hat{\lambda}$ the mode with $m = 0$ has the largest imaginary part, but a larger $\hat{\lambda}$ can lead to modes with higher m being dominant.

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Table 1: Parameters Used in `elegant` Simulations

Parameter	Symbol	Value
Vertical tune	ν_y	36.1
Chromaticity	ξ_y	0 to 5.5
Momentum compaction	α_c	4.04×10^{-5}
Bunch length	σ_z	16.06 mm
Energy spread	σ_δ	0.135%
Synchrotron frequency	$\omega_s/2\pi$	160 Hz
Chromatic phase	$k_\xi \sigma_z$	0 to 12.5
Number of bunches	N_b	48
Total current	I_{tot}	200 mA
Vertical damping time	τ_y	15.4 ms
Coupled-bunch eigenvalue	λ	$\Im(\lambda)(i - 0.6)$

On the other hand, when the instability growth rate becomes of order the synchrotron frequency, $\Im(\hat{\Omega}) > 1$ one can show that

$$1 = \frac{-i\hat{\lambda}}{k_\xi \sigma_z}\sqrt{\frac{\pi}{2}}e^{-\hat{\Omega}^2/2k_\xi^2\sigma_z^2}\text{erfc}\left(\frac{-i\hat{\Omega}}{\sqrt{2}k_\xi\sigma_z}\right). \quad (14)$$

If $|\Re(\lambda)| \leq \Im(\lambda)$ as is typical, we find that Eq. (14) predicts that $\Im(\hat{\Omega}) = 0$ when the instability strength $\hat{\lambda} \approx (0.74 \pm 0.06)k_\xi\sigma_z$. Increasing $\Im(\hat{\lambda})$ beyond this range results in a similar (or larger) increase in the coupled-bunch instability growth rate, and the reduced slope given by (13) no longer applies. Hence, we find that the chromaticity is only effective in controlling the instability provided that

$$\Im(\lambda) < \frac{3k_\xi\sigma_z}{4}\frac{\alpha_c\sigma_\delta}{\sigma_t} = \frac{3}{4}\xi\omega_0\sigma_\delta. \quad (15)$$

Hence, the coupled-bunch growth rate is reduced according to Eq. (13) if the spread in the betatron frequency due to chromatic effects $\xi\omega_0\sigma_\delta$ is much larger than the $\xi = 0$ growth rate λ, while if $\Im(\lambda) > \xi\omega_0\sigma_\delta$ Eq. (15) applies and $\Im(\Omega)$ becomes significantly larger than that implied by (13).

We plot solutions to Eq. (12) when $\Re(\lambda) = \Im(\lambda)$ in Fig. 1(a). When $\xi \neq 0$ two specific regimes can be identified: the first applies when $\Im(\hat{\lambda}) < 3k_\xi\sigma_z/4$, and displays an instability growth rate $\Im(\hat{\Omega})$ that increases with $\Im(\hat{\lambda})$ at a rate inversely proportional to $k_\xi\sigma_z$; the second regime takes over when $\Im(\hat{\lambda}) > 3k_\xi\sigma_z/4$, and predicts that the slope of the growth rate with $\Im(\hat{\lambda})$ is greater than (but comparable to) that for $\xi = 0$.

We also compare solutions of (12) to those obtained from `elegant` tracking [9] in Fig. 1(b). For this comparison we use lattice parameters relevant to the APS-U storage ring that are listed in Table 1, and we ignore synchrotron emission. In particular, note that the strength of the matrix growth rate λ is varied while maintaining the ratio $\Re(\lambda)/\Im(\lambda)$ derived for the long-range resistive wall wakefield assuming a fractional tune of 0.1 and the 48 equi-spaced bunch pattern, and the bunch length was chosen to match the rms σ_z of the double rf system planned for APS-U.

Figure 1(b) shows that our theory agrees very well with tracking over the entire range of λ and ξ. In particular, both

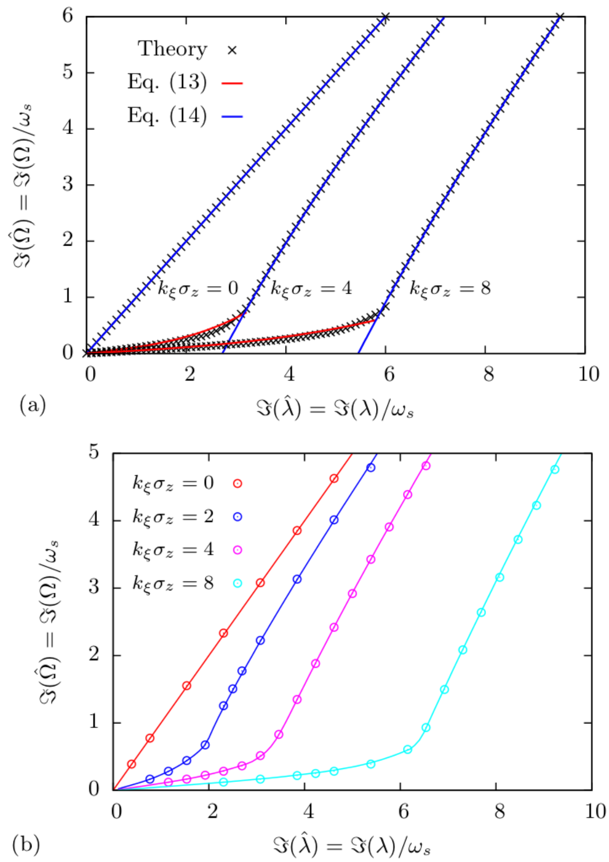

Figure 1: (a) Theoretical growth rates (crosses) with the weak limit Eq. (13) in red and the strong limit Eq. (14) in blue. (b) Comparison of the growth rates obtained by tracking (dots) to those of theory (solid lines) when $V_z \propto z^2$ and we assume no synchrotron damping.

show clear evidence of the weak instability regime for small λ, and the strong regime when Eq. (15) applies.

CONCLUSIONS

We have sketched how to derive the dispersion relation (12) that relates the complex growth rate Ω of multi-bunch transverse stability to the $\xi = 0$ growth rate λ and the head-tail (chromatic) phase shift over the bunch $k_\xi\sigma_z$. The relation predicts two regimes of the instability depending upon whether the $\xi = 0$ growth rate is smaller or larger than the chromatic tune shift over the bunch, and agrees well with simulations when synchrotron radiation is ignored. Future work will explore similar results for a longitudinal potential that is a quartic function of z as might be the case for a double rf system, and will also compare theory to simulations that include synchrotron emission.

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