ISBN: 978-3-95450-232-5 ISSN: 2673-7000 doi:10.18429/JACoW-NAPAC2022-WEPA34

TRANSFER MAPS IN THE HARD-EDGE LIMIT OF QUADRUPOLE AND BEND MAGNETS FRINGE FIELDS*

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Abstract

Beam dynamics of charged particles in the fringe field of a quadrupole and a dipole magnet is considered. An effective method for solving symplectic Lie map $\exp(:f:)$ in such cases has been developed. A precise analytic solution for nonlinear transverse beam dynamics in a quadrupole magnet with hard-edge fringe field has been obtained. The method of Lie map calculation considered here can be applied for other magnets and for soft edge type of fringe field.

INTRODUCTION

Nonlinear beam dynamics due to the fringe field effect of magnets can impact accelerator performance [1, 2]. In this paper we calculated an exact analytic nonlinear transfer map for beam dynamics in the fringe field of a quadrupole magnet using the Lie transformation $\exp(:f:)$ in hard-edge approximation. In accelerator physics, transverse beam optics of single particle is defined by 4D coordinates in a phase space $\{x, x_p, y, y_p\}$. Here $x_p = p_x/p_z$ and $y_p = p_y/p_z$ are angular coordinates. Solving the Hamiltonian flow with input coordinates $\{x, x_p, y, y_p\}_{in}$ gives a corresponding function $\mathbf{M}$ called a transfer map with output coordinates $\{x, x_p, y, y_p\}_{out} = \mathbf{M}\{x, x_p, y, y_p\}_{in}$.

The beam dynamics for real quadrupole magnets is normally calculated numerically and the transfer maps is hard to solve precisely and analytically in the general case. For this reason, the magnetic field of the quadrupole as well as the beam dynamics is usually simplified by a so-called hard-edge model. The real 3D magnetic field of a quadrupole $\mathbf{B}(x, y, z)$ can be defined by its gradient components $G(z) = \partial B_y/\partial x = -\partial B_x/\partial y$ on the z-axis. The hard-edge model is represented by a rectangular profile with amplitude $G = G(z)_{max}$ and longitudinal size $L_{eff} = \int G(z)dz/G$ [3]. Single particle dynamics is well approximated by the hard-edge model consisting of a sequence of elements called lattice: drift space followed by nonlinear thin element of the entrance fringe field followed by thick linear element of the quadrupole magnet [3] then followed by thin nonlinear element of the exit fringe field. In this paper we consider the nonlinear beam dynamics of the entrance and exit fringe fields of the quadrupole and dipole. The nonlinear map in first order approximation over $\mathbf{B}$ field gradient was originally derived by Lee-Whiting [4] in 1970. Some derivations for electrostatic and magnetic quadrupole lens can be found in [5]. Linear map due to fringe field can be found in [6].

The nonlinear transfer map can be obtained by calculating Lie transformation widely used in accelerator physics [7, 8]:

$$\exp(:f:) = \sum_{n=0}^{\infty} \frac{:f:^n}{n!} \tag{1}$$

Here $:f:^n$ is operator defined by recurrence equation:

$$:f:^n = \{f, :f:^{n-1}\} \tag{2}$$

with the initial condition $:f:^0 z_i = z_i$. Here z_i is one of the phase space coordinate to be transformed by Eq. (1) and

$$\{a, b\} = \frac{\partial a}{\partial x}\frac{\partial b}{\partial x_p} - \frac{\partial a}{\partial x_p}\frac{\partial b}{\partial x} + \frac{\partial a}{\partial y}\frac{\partial b}{\partial y_p} - \frac{\partial a}{\partial y_p}\frac{\partial b}{\partial y} \tag{3}$$

is the Poisson bracket operator expressed in terms of transverse phase space coordinates. f in expression Eq. (1) is called characteristic function [7, 9]. We will derive a precise nonlinear map for the entrance/exit fringe fields of quadrupole and dipole magnets by solving the Lie transformation Eq. (1) exactly.

QUADRUPOLE MAGNET

We will use magnetic field and its corresponding characteristic function f in the form of multipole approximation [7, 9]:

$$f = \mathbf{Re}\frac{C(x+iy)^n}{4(n+1)}\left(xp_x + yp_y + i\frac{n+2}{n}(xp_y - yp_x)\right) \tag{4}$$

This field is characterised by the lowest order harmonic n in the transverse direction while any real field contains an infinite number of harmonics according to Fourier theory.

In this section we solve precisely Lie transformation Eq. (1) with characteristic function Eq. (4) for the fringe field of quadrupole with $n = 2$:

$$\exp(:k[y_p(y^3 + 3x^2y) - x_p(x^3 + 3y^2x)]:) \tag{5}$$

Here we denote the strength coefficient

$$k = \frac{eG/p_0}{12(1 + \delta p_0/p_0)} \tag{6}$$

for simplicity, where G is the amplitude of the gradient of the magnetic field. Entrance and exit fringe fields are characterised by ± sign in front of k coefficient while the sign of the coefficient k can be positive or negative independently. Before we solve the full transformation Eq. (1) with four phase-space terms of the characteristic function f we will first solve the Lie transform for only one term:

$$\exp(:-kx_px^3:) \tag{7}$$

* This manuscript has been authored by UT-Battelle, LLC, under Contract No. DE-AC05-00OR22725 with the U.S. Department of Energy.

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ISBN: 978-3-95450-232-5 ISSN: 2673-7000 doi:10.18429/JACoW-NAPAC2022-WEPA34

After applying Eq. (7) to x we obtain the infinite series:

$$e^{:-kx_p x^3:}x = x + \frac{kx^3}{1!} + \frac{3k^2x^5}{2!} + \ldots + \frac{a_n k^n x^{2n+1}}{n!} + \ldots \quad (8)$$

The numerical coefficient a_n in Eq. (8) can be calculated by substituting $a_n k^n x^{2n+1}$ into Eq. (2). After transformations we obtain:

$$a_n = (2n-1)a_{n-1} \quad (9)$$

which can be solved with the initial condition $a_0 = 1$:

$$a_n = \frac{(2n)!}{2^n n!} \quad (10)$$

Using this, precise transfer map Eq. (8) for $x^f \equiv x_{out}$ after infinite series summation equals:

$$e^{:-kx_p x^3:}x = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n (n!)^2} k^n x^{2n+1} = \frac{x}{\sqrt{1-2kx^2}} \quad (11)$$

Similar calculation can be done for other canonical coordinates $\{x, x_p, y, y_p\}$. The transfer map Eq. (8) is easily solvable but it is much more complicated with increasing number of terms of the characteristic function Eq. (5). In this case it would be helpful to know or assume a general algebraic form of solution to be found. After that, we can calculate a solution in the given form by applying a simplifying transformation $F[z]$ to the map itself and using the following rule:

$$F[\exp(:f:)z] = \exp(:f:)F[z] \quad (12)$$

Considering problem Eq. (7) it would be convenient to reduce the square root from solution Eq. (11) and turn over the whole fraction applying the total transformation $F[z] = z^{-2}$. Then we can solve it in a simple way without series calculation:

$$(e^{:-kx_p x^3:}x)^{-2} = e^{:-kx_p x^3:}(x^{-2}) = \frac{1}{x^2} - 2k \quad (13)$$

Solution Eq. (13) must be transformed back $(z^f)^{-1/2}$ in order to get the result of Eq. (11). The simplification Eq. (13) works well for any number of terms in Eq. (5) and we will present the full solution of Eq. (5) without detailed solving:

$$\begin{aligned}
x^f &= \frac{x + y + (x-y)\sqrt{\cos 2\phi} + \dfrac{2xy}{x+y}\left(\cos 2\phi + \dfrac{\sqrt{x^2+y^2}}{\sqrt{2xy}}\sin 2\phi - 1\right)}{\sqrt{\dfrac{4\sqrt{\cos 2\phi}}{(x+y)^2}\left((x^2+y^2)\cos 2\phi + 2xy\right) - 2\sin 2\phi\,\dfrac{x-y}{x+y}\left(\dfrac{\sqrt{x^2+y^2}}{\sqrt{2xy}} - \dfrac{\sqrt{2xy}}{\sqrt{x^2+y^2}}\cos 2\phi\right)}} \\
y^f &= \frac{x + y - (x-y)\sqrt{\cos 2\phi} + \dfrac{2xy}{x+y}\left(\cos 2\phi - \dfrac{\sqrt{x^2+y^2}}{\sqrt{2xy}}\sin 2\phi - 1\right)}{\sqrt{\dfrac{4\sqrt{\cos 2\phi}}{(x+y)^2}\left((x^2+y^2)\cos 2\phi + 2xy\right) - 2\sin 2\phi\,\dfrac{x-y}{x+y}\left(\dfrac{\sqrt{x^2+y^2}}{\sqrt{2xy}} - \dfrac{\sqrt{2xy}}{\sqrt{x^2+y^2}}\cos 2\phi\right)}} \quad (14) \\
x_p^f &= \left(x_p\frac{\partial y^f}{\partial y} - y_p\frac{\partial y^f}{\partial x}\right) \Big/ \left(\frac{\partial y^f}{\partial y}\frac{\partial x^f}{\partial x} - \frac{\partial x^f}{\partial y}\frac{\partial y^f}{\partial x}\right) \\
y_p^f &= \left(y_p\frac{\partial x^f}{\partial x} - x_p\frac{\partial x^f}{\partial y}\right) \Big/ \left(\frac{\partial y^f}{\partial y}\frac{\partial x^f}{\partial x} - \frac{\partial x^f}{\partial y}\frac{\partial y^f}{\partial x}\right) \\
\phi &= \mathrm{am}\left(\pm 2k\sqrt{2xy}\sqrt{x^2+y^2}\,\middle|\,2\right)
\end{aligned}$$

Here the $\pm$ sign refers to the entrance/exit fringe field maps and $\phi = \mathrm{am}(z\,|\,2)$ is the Jacobi Amplitude function defined as the inverse of the elliptic integral:

$$z = \int_0^{\phi} \frac{dt}{\sqrt{1-2\sin^2 t}} \quad (15)$$

Function $\mathrm{am}(z\,|\,2)$ has a series representation:

$$\mathrm{am}(z\,|\,2) = z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{3z^7}{70} + \ldots \quad (16)$$

The canonical momenta x_p^f and y_p^f are presented through derivative of functions x^f and y^f because they are too bulk to present in explicit form. This representation follows from symplecticic property of the Lie transformation Eq. (1). The parameters x_p^f, y_p^f in Eq. (14) can be easily calculated and represented through the $\phi = \mathrm{am}(z\,|\,2)$ parameter using a

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derivative transformation:

$$\frac{\partial}{\partial z}\mathrm{am}(z\,|\,2) = -\sqrt{\cos\left(2\,\mathrm{am}(z\,|\,2)\right)} = -\sqrt{\cos 2\phi} \qquad (17)$$

Solution Eq. (14) can be verified by Taylor series expansion in the k coefficient and compared with Eq. (1) series.

DIPOLE MAGNET

Here we present a transfer map for the fringe field of a dipole magnet in the hard edge approximation. After substituting $n = 1$ into Eq. (4) the Lie transformation for the fringe field has the following form:

$$\exp(:\,k[2y_p xy - x_p(x^2+3y^2)]\,:) \qquad (18)$$

with coefficient

$$k = \frac{eB_0/p_0}{8(1+\delta p_0/p_0)} \qquad (19)$$

where B_0 is the magnetic field amplitude of the dipole magnet. Solution Eq. (18) for a dipole magnet has the following form:

$$\begin{aligned}
x^f &= \frac{\pm\frac{1}{k^3}\sqrt{k^6 g^3 - k^6 y^2\left(x^2+y^2\right)^2}\left(g(x^2+3y^2)+6y^2(x^2+y^2)\right) + xg^3 + 2xy^2(x^2+y^2)(3g+x^2+y^2)}{\left(g-x^2-y^2\right)\left(g^2+4y^2(g+x^2+y^2)\right)}\\
y^f &= \frac{y\left(g-x^2-y^2\right)^2}{\pm\frac{2x}{k^3}\sqrt{k^6 g^3 - k^6 y^2\left(x^2+y^2\right)^2} + g^2 + (x^2+y^2)(g-2y^2)}\\
x_p^f &= \left(x_p\frac{\partial y^f}{\partial y} - y_p\frac{\partial y^f}{\partial x}\right)\Big/\left(\frac{\partial y^f}{\partial y}\frac{\partial x^f}{\partial x} - \frac{\partial x^f}{\partial y}\frac{\partial y^f}{\partial x}\right)\\
y_p^f &= \left(y_p\frac{\partial x^f}{\partial x} - x_p\frac{\partial x^f}{\partial y}\right)\Big/\left(\frac{\partial y^f}{\partial y}\frac{\partial x^f}{\partial x} - \frac{\partial x^f}{\partial y}\frac{\partial y^f}{\partial x}\right)\\
g &= \left(yx^2+y^3\right)^{2/3}\left(1+\frac{1+\cos\phi}{1-\cos\phi}\sqrt{3}\right), \qquad \phi = \mathrm{am}\left(2\times 3^{1/4}k\left(yx^2+y^3\right)^{1/3}\,\Big|\,\tfrac{2-\sqrt{3}}{4}\right)
\end{aligned} \qquad (20)$$

Here the $\pm$ sign refers to the entrance/exit fringe field maps, $\phi = \mathrm{am}(z\,|\,(2-\sqrt{3})/4)$ is the Jacobi Amplitude function defined as the inverse of the elliptic integral:

$$z = \int_0^\phi \frac{dt}{\sqrt{1-\frac{2-\sqrt{3}}{4}\sin^2 t}} \qquad (21)$$

Expressions for angular phase space coordinates x_p^f, y_p^f are written in the same way as in Eq. (14) for simplicity and can be easily calculated using the transformation for the Jacobi Amplitude derivative:

$$\frac{\partial}{\partial z}\mathrm{am}\left(z\,\Big|\,\tfrac{2-\sqrt{3}}{4}\right) = \frac{1}{2}\sqrt{2+\sqrt{3}+(2-\sqrt{3})\cos^2\phi} \qquad (22)$$

Solution Eq. (20) as well as Eq. (14) can be verified by Taylor series expansion in the k parameter and compared with the Lie series expansion Eq. (1).

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