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PROGRESS ON CONVERGENCE MAP BASED ON SQUARE MATRIX FOR NONLINEAR LATTICE OPTIMIZATION*

L. H. Yu†, Y. Hidaka, F. Plassard, V. Smaluk, Brookhaven National Laboratory, Upton, NY, USA
Y. Hao, Michigan State University, East Lansing, MI, USA

Abstract

We report progress on applying the square matrix method to obtain in high speed a "convergence map", which is similar but different from a frequency map. We give an example of applying the method to optimize nonlinear lattice for NSLS-II. The convergence map is obtained from solving nonlinear dynamic equations by iteration of the perturbation method and studying the convergence. The map provides information about the stability border of the dynamic aperture. We compare the map with the frequency map from tracking. The result in our example of nonlinear optimization of NSLS-II lattice shows the new method may be applied in nonlinear lattice optimization, taking the advantage of the high speed (about 30 to 300 times faster) to explore horizontal, vertical, and the off-momentum phase space.

INTRODUCTION

A study of the long-term behavior of charged particles in storage rings is one of the topical applications of nonlinear dynamics. The analysis of the particle behavior is based on many iterations of the particle phase space transformation by the one-turn map representing the storage ring. The most accurate and reliable numerical approach is particle tracking in a magnet lattice model with appropriate integration methods. This approach is implemented in many computer codes. However, particle tracking is very demanding to computing resources, so parallel codes and long computation time are often required.

For fast analysis, however, one would like a more compact representation of the one-turn map out of which to extract relevant information. Among many approaches to this issue, we may mention canonical perturbation theory, Lie operators, power series, normal form, etc. [1–3]. The results are often expressed as polynomials. However, for increased perturbation, near resonance or for large oscillation amplitudes, these perturbative approaches often have insufficient precision. The stability analysis of the beam trajectory and calculation of the dynamic aperture requires an accurate solution of the nonlinear dynamic equation. Hence there is a need to extract information about long-term particle behavior from the one-turn map based on these polynomials with high precision and high speed.

The square matrix analysis [4] has a good potential to explore this area. On a basis of the square matrix method, we developed a novel technique of "convergence map", which is a much faster alternative to the tracking-based frequency map [5]. The convergence map provides information about the dynamic aperture and can be applied to nonlinear lattice optimization, taking the advantage of the high speed (30 to 300 times faster than particle tracking). The computation speed ratio is larger for complex lattices with low periodicity, such as particle colliders.

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† email address: lhyu@bnl.gov

SQUARE MATRIX METHOD FOR ANALYSIS OF NONLINEAR DYNAMICS

A novel method to analyze nonlinear dynamic systems using the square matrix has been developed at NSLS-II few years ago [4]. We showed that for a nonlinear dynamic system representing particle motion in a storage ring, we can construct a square matrix. Using linear algebra, the Jordan decomposition of the square matrix provides a tool for studying the fluctuation of particle oscillation frequency, the stability of the particle trajectories, and dynamic aperture. Thus, the analysis of a nonlinear dynamic system can be greatly simplified using linear algebra. The square matrix method is general and may be applied to other areas, for example, nonlinear dynamics in physics and astronomy.

The main feature of the new method is that we can achieve high order in one step. This is a significant advantage when compared with canonical perturbation theory and normal form, where the calculation is carried out order-to-order by a complicated iteration process. We also showed that the stability and precision of the Jordan decomposition are ensured by scaling the variables, and by removing the high-power invariant monomial terms.

We demonstrated that the action variable remains nearly constant up to near the boundary of the dynamic aperture and resonance lines. They successfully reproduce both the correct phase space structure and the betatron tune shift with amplitude. In addition, we tested several measures of the stability of particle trajectories and their betatron tunes.

The developed theory shows good potential in theoretical understanding of a complex dynamic system to guide the optimization of dynamic aperture in circular accelerators. Using analysis of the one-turn map to narrow down the searching range of the parameter space before the final confirmation by tracking, the new method can significantly speed up the optimization.

CONVERGENCE MAP VS PARTICLE TRACKING

We introduce a convergence map calculated using action-angle variables in the form of polynomials provided by a square matrix, which is derived from the one-turn map for an accelerator lattice. Since the iterations leading to the solution of the nonlinear dynamic equations expressed by these action-angle variables can be carried out by Fourier transform, the computation speed is very high. Using the

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NSLS-II storage ring lattice [6] as an example, we show the nonlinear lattice optimization using the convergence map results in a dynamic aperture comparable to or larger than that obtained by particle tracking but the calculations are much faster. The NSLS-II storage ring lattice consists of 15 super-periods, Twiss functions of one super-period (two cells, 7th and 8th) are shown in Figure 1.

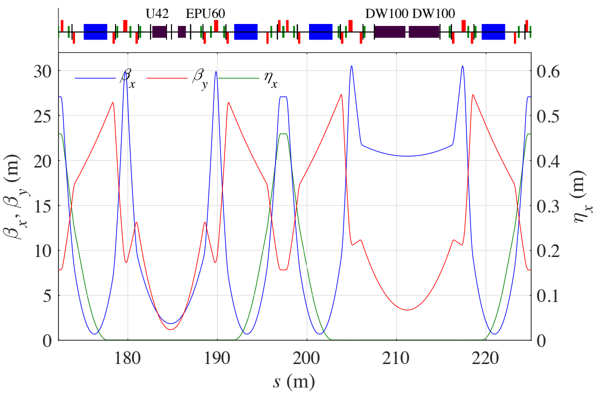

Figure 1: One super-period of the NSLS-II lattice.

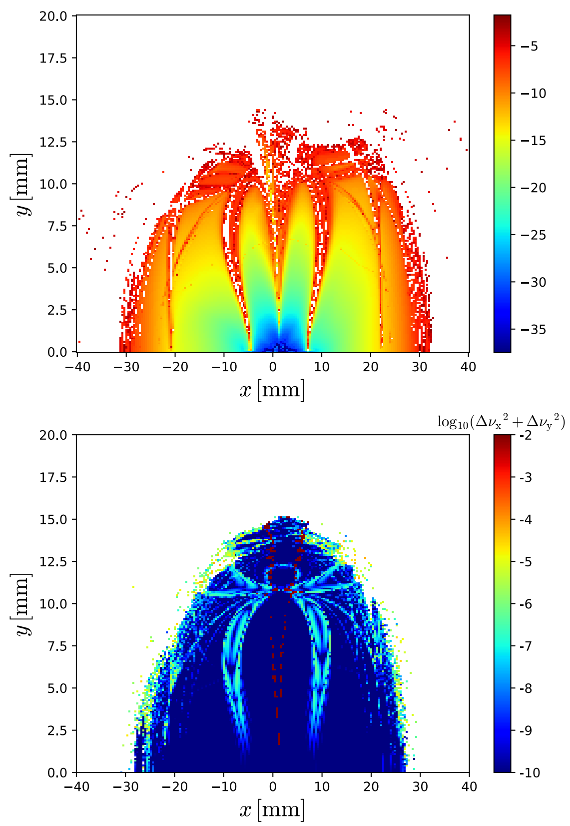

Figure 2: Convergence map (top) vs frequency map from tracking (bottom).

We compared convergence map with the frequency map [5] calculated by particle tracking. For the nominal NSLS-II lattice, Figure 2 shows the convergence map (a) and the frequency map (b) calculated using ELEGANT tracking code [7] with the same number of points in horizontal (x) and vertical (y) planes. For the convergence map, the computation time is about 300 times shorter for this case.

As one can see, both diagrams show the same stable area and major resonances. Thus, the convergence map provides the same information about the nonlinear motion and dynamic aperture, but the computation time is shorter by a factor of 30 to 300 depending on the size and order of symmetry of the ring lattice (number of super-periods), as shown in the next section.

BENCHMARKING OF COMPUTATION TIME

Computation time was compared between the convergence map and the frequency map using one super-period (2 cells) and the whole ring (15 super-periods) of the NSLS-II bare lattice. By "bare", it means there is no insertion device element in the lattice.

To compare the two maps, we need to compare the computation time for selected points in x-y plane. If we choose the points in an unstable region, some particles may be lost during tracking, this would make the comparison difficult. Hence, we need to choose these points in a specially specified stable region, and as we use a different number of points for comparison, we need the points confined within the specified region. For both types of maps, an initial coordinate region of $+10 \le x[\mathrm{mm}] \le +11$ and $+1 \le y[\mathrm{mm}] \le +2$ was selected as particles launched from this region are very stable and can last at least 1024 turns specified for frequency map analysis. This square region was divided into 2×2, 3×3, 5×5, 10×10, 50×50, 100×100 grid points. Each grid point is used as an initial transverse coordinate for both maps. The momentum offset was zero.

For frequency map computations, we used ELEGANT's "frequency map" command [7] to compute the diffusion defined by the tune changes between the first 512 and the latter 512 turns.

For convergence map computations, PyTPSA [8] was used to create truncated power series (TPS) [2,3] objects and handle all the algebraic operations on them while the TPS objects propagated through all the lattice elements in a Python module where the symplectic integration method of TRACY [9] has been reimplemented. The polynomials based on Jordan form are of the 3rd power order.

All the computations were performed using a single core of Intel Xeon Gold 6252 CPU at 2.10 GHz (hyper-threading enabled). Figure 3 represents the computation time of the convergence map and the frequency map as a function of the number of points in both planes for one super-period (1SP) and the whole NSLS-II ring (15SP).

The computation time of frequency maps (FM) is linearly scaled with the number of grid points as expected. It was also expected to linearly scale with the number of super-periods (SP), as each point requires tracking of a single particle from the beginning to the end of the selected lattice. Thus, the whole-ring lattice should have taken roughly 15 times longer than the 1-SP lattice. However, the time only increased by 10.5 times. This appears to indicate the overhead of the non-tracking portion of ELEGANT's code is not negligible, compared to the tracking portion.

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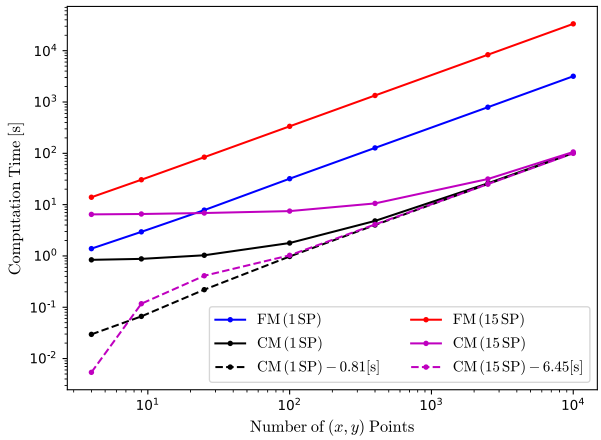

Figure 3: Computation time comparison between convergence map (CM) and frequency map (FM) analyses with different numbers of initial transverse coordinate points and with one super-period (SP) and the whole ring (15 SPs) of NSLS-II. The dash lines correspond to the computation times without the initial setup times (mainly TPSA calculations).

The most notable feature of the convergence map (CM) time is the fact that it changed very little for the case of 10^4 points whether the lattice was of 1 or 15 super-periods. This makes sense because once the TPSA calculation for a lattice is finished at the beginning, the computation cost is the same for each grid point, whether the lattice was of 1 or 15 super-periods, unlike the tracking-based FM whose computation time is proportional to the length of the lattice. Note that the initial TPSA calculation does depend on the length of the lattice. However, it only increased from 0.81 s for 1 SP to 6.45 s for 15 SPs. In both cases, this initial setup time is tiny compared to the total time of 100 seconds it took to compute the convergence values for 10^4 points.

The speed of CM for 10^4 points was 31.2 times faster than that of FM for the 1-SP case, while it was 314 times faster for the 15-SP case. These speed improvement factors include all the overhead and initial setup times. However, the advantage of CM diminishes as the number of points decrease, since the initial TPSA computation time starts to dominate the total CM computation time. Therefore, CM is particularly useful when the number of initial coordinate points whose stability needs to be investigated is quite large and/or when the lattice under study is very long and complex (e.g., lattices with multipole and alignment errors included and lattices with no periodicity such as colliders).

The dashed lines in Figure 3 shows the computation times for CM without the initial setup times. Both the 1-SP and 15-SP curves show good linearity with the number of grid points. They are also almost on top of each other. This demonstrates the earlier statement of the convergence value computation time being independent of the lattice length/complexity, as long as the initial TPSA computation time is excluded.

CONCLUSION

We show that the evolution of action-angle variables derived from the square matrix method is close to a pure rotation, hence it is possible to rewrite the nonlinear dynamic equations in terms of these variables as an exact equation. The equations are in the form of pure rotation with nonlinear terms as a perturbation. Hence iteration steps developed using the perturbation method to solve the nonlinear dynamic equation are convergent up to the dynamic aperture or the border of a resonance region. The convergence rate varies depending on how the trajectory is close to the dynamic aperture or the resonance region. Hence the convergence rate is a function of the phase space. For example, the convergence rate can be plotted over x-y plane using a color scale. This convergence map can be used to study the stability of a nonlinear dynamic system.

This convergence map looks similar but is very different from the frequency map calculated by particle tracking. The dynamic aperture, tune footprint, phase space trajectory, and frequency spectrum calculated using the convergence map agree with the tracking to high precision.

However, the convergence map method is much faster than tracking, hence it can be used for nonlinear lattice optimization. Using the NSLS-II lattice as an example, we carried out an extensive comparison of the optimization by the traditional tracking method with the convergence map. We compared the speed and the quality of the optimization and show that depending on the complexity of the lattices, the speed of the convergence map method is 30 to 300 times faster. Hence, we demonstrated that the convergence map is an efficient tool for nonlinear optimization, especially for complex lattices with low or no periodicity.

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