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EXTENDED SOFT-GAUSSIAN CODE FOR BEAM-BEAM SIMULATIONS*

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Abstract

Large ion beam emittance growth is observed in strong-strong beam-beam simulations for the Electron-Ion Collider (EIC). As we know, the Particle-In-Cell solver is subject to numerical noises. As an alternative approach, an extended soft-Gaussian code is developed with help of Hermite polynomials in this paper. The correlation between the horizontal and the vertical coordinates of macro-particles is considered. The 3rd order center moments are also included in the beam-beam force. This code could be used as a cross check tool of PIC based strong-strong simulation.

INTRODUCTION

The beam-beam interaction is one of the most important phenomena to limit the luminosity in colliders. Beam-beam simulation is an essential tool to study beam-beam effects. Two models are often used in simulations: weak-strong and strong-strong. The weak-strong model is used to study the single particle dynamics, while the strong-strong model is used to study the coherent motion.

The particle-in-cell (PIC) approach is widely used in strong-strong simulation. This kind of method is self-consistent because the electromagnetic field is obtained by solving the Poisson equation with the updated charge distribution during beam collisions [1]. However, the PIC based strong-strong simulation is subject to numerical noise. The discrepancy between the weak-strong and strong-strong simulation for Electron-Ion Collider (EIC) has been found. It is important to understand the difference in case there is some coherent mechanism shadowed by the large numerical noise.

The soft-Gaussian model is a possible way to cross-check the PIC results. In the soft-Gaussian model, both beams are assumed to be an ideal Gaussian distribution during the collision. The second-order moments σ_x and σ_y are calculated from macro particles. Although the soft-Gaussian model is not self-consistent, the coherent motion is considered during the collision.

However, there is a possibility to over-simplify the problem in the soft-Gaussian model. In this paper, we develop a code to extend the soft-Gaussian model. In the extended soft-Gaussian model (ESG), the 3rd order moments are also taken into account in the calculation of electromagnetic field. The ESG would be a better benchmark tool for strong-strong simulation.

INCLUDING BEAM TILT

The beam-beam potential generated by an upright bi-Gaussian distribution is

$$U_g = \frac{Q_1 Q_2 N r_0}{\gamma_0} \int_0^\infty \mathrm{d}u \frac{\exp\left(-\frac{x^2}{2\sigma_x^2+u} - \frac{y^2}{2\sigma_y^2+u}\right)}{\sqrt{2\sigma_x^2+u}\sqrt{2\sigma_y^2+u}}, \quad (1)$$

where N is the total particle number, $r_0 = e^2/(4\pi\epsilon_0 mc^2)$ the classical radius, γ_0 the relativistic factor of the test particle, $Q_{1,2}$ the charge numbers of particles from two colliding bunches, and $\sigma_{x,y}$ are the RMS beam sizes at the collision point.

The deflection angle from the above bi-Gaussian beam can be obtained from the well-known Bassetti-Erskine formula [2],

$$\begin{aligned} U_y + \mathrm{i}U_x = &-\frac{Q_1 Q_2 N r_0}{\gamma_0}\sqrt{\frac{2\pi}{\sigma_x^2-\sigma_y^2}}\left[w\left(\frac{x+\mathrm{i}y}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right)\right. \\ &\left.- w\left(\frac{\frac{\sigma_y}{\sigma_x}x+\mathrm{i}\frac{\sigma_x}{\sigma_y}y}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right)\exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)\right] \end{aligned} \quad (2)$$

where $U_{x,y}$ is the abbreviation of the derivative $\partial U_g/\partial y$ or $\partial U_g/\partial x$, x, y the coordinates of the test particle, and $w(z)$ is the Faddeeva function,

$$w(z) \equiv \exp\left(-z^2\right)\left(1+\frac{2\mathrm{i}}{\sqrt{\pi}}\int_0^z \mathrm{d}t\, e^{t^2}\right) \quad (3)$$

In the long term tracking, both beams may tilt slowly in the $x-y$ planes because of the nonlinear coupling of beam-beam force. As a result, the non-zero σ_{xy} should be considered for more accurate calculation.

A general 2D Gaussian distribution can be described by its Σ matrix,

$$\Sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \quad (4)$$

$$\phi_g(x,y) = \frac{1}{2\pi\sqrt{\det\Sigma}}\exp\left[-\frac{1}{2}(x,y)\,\Sigma^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right], \quad (5)$$

where $\phi_g(x,y)$ is the 2D distribution in (x,y) plane. To use the Bassetti-Erskine formula Eq. (2), we can apply a rotation on the coordinates (x,y),

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \qquad \begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix} = A\begin{bmatrix}x\\y\end{bmatrix}, \quad (6)$$

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so that the Σ matrix in the rotated frame is diagonal:

$$\begin{bmatrix} \sigma_{yy} & -\sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{bmatrix} = A^{\mathrm{T}} \begin{bmatrix} \overline{\sigma}_{yy} & 0 \\ 0 & \overline{\sigma}_{xx} \end{bmatrix} A \,, \tag{7}$$

where the overline denotes the variable in the rotated frame.

A possible solution is:

$$\cos 2\theta = \frac{\sigma_{xx} - \sigma_{yy}}{\sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4\sigma_{xy}^2}} \tag{8}$$

$$\sin 2\theta = \frac{2\sigma_{xy}}{\sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4\sigma_{xy}^2}} \tag{9}$$

$$\overline{\sigma}_{xx} = \frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right) + \frac{1}{2}\sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4\sigma_{xy}^2} \tag{10}$$

$$\overline{\sigma}_{yy} = \frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right) - \frac{1}{2}\sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4\sigma_{xy}^2} \,. \tag{11}$$

Rotating back to the original $x - y$ frame, the deflection angle by beam-beam interaction is:

$$\begin{bmatrix} U_x \\ U_y \end{bmatrix} = A^{-1} \begin{bmatrix} \overline{U}_x \\ \overline{U}_y \end{bmatrix} \,, \tag{12}$$

where $\overline{U}_{x,y}$ is calculated from Eq. (2) with substitution of $\overline{x}, \overline{y}, \overline{\sigma}_{x,y}$.

INCLUDING THIRD-ORDER MOMENTS

After the frame is tilted, we can further extend the model to include higher order moments with the help of Hermite polynomial [3],

$$\phi\left(x, y\right) = a_{ij} H_i\left(x/\sigma_x\right) H_j\left(y/\sigma_y\right) \phi_g(x, y) \tag{13}$$

where the repeated indices mean summation, $\phi_g\left(x, y\right)$ the standard bi-Gaussian kernel as shown in Eq. (5). The Hermite polynomial is defined as

$$H_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right) \frac{\mathrm{d}^n}{\mathrm{d}x^n} \exp\left(-\frac{x^2}{2}\right) \,. \tag{14}$$

Because of the orthogonality,

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) \exp\left(-\frac{x^2}{2}\right) \,\mathrm{d}x = \sqrt{2\pi} n! \delta_{mn} \,, \tag{15}$$

and the coefficient is determined by

$$\begin{aligned} a_{mn} &= \frac{1}{m!n!} \int H_m\left(x/\sigma_x\right) H_n\left(y/\sigma_y\right) \phi\left(x, y\right) \,\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{m!n!} \left\langle H_m\left(x/\sigma_x\right) H_n\left(y/\sigma_y\right) \right\rangle \,, \end{aligned} \tag{16}$$

where the angle bracket means taking the average over all macro particles in simulation. Then the corresponding beam-beam potential is given by:

$$U = (-1)^{m+n} a_{mn} \sigma_x^m \sigma_y^n \frac{\partial^m}{\partial x^m} \frac{\partial^n}{\partial y^n} U_g \,. \tag{17}$$

To save computation time, we only extend the model to include the third-order moments. In the rotated frame, the first two orders are corrected zero,

$$\begin{aligned} & a_{00} = 1, \quad a_{10} = a_{01} = a_{20} = a_{11} = a_{02} = 0 \\ & a_{30} = \frac{1}{6}\left\langle \frac{x^3}{\sigma_x^3} \right\rangle, \qquad a_{21} = \frac{1}{2}\left\langle \frac{x^2 y}{\sigma_x^2 \sigma_y} \right\rangle \\ & a_{03} = \frac{1}{2}\left\langle \frac{x y^2}{\sigma_x \sigma_y^2} \right\rangle, \qquad a_{03} = \frac{1}{6}\left\langle \frac{y^3}{\sigma_y^3} \right\rangle \,. \end{aligned} \tag{18}$$

Up to third order, the beam-beam potential is

$$\begin{aligned} U = U_g &- a_{30}\sigma_x^3 U_{xxx} - a_{21}\sigma_x^2\sigma_y U_{xxy} \\ &- a_{12}\sigma_x\sigma_y^2 U_{xyy} - a_{03}\sigma_y^3 U_{yyy} \end{aligned} \tag{19}$$

where $U_{xxx}, U_{xxy}, U_{xyy}, U_{yyy}$ on the right hand are the partial derivatives of U_g. The analytic expression can be found in [4].

In our code, there are 20 terms of third order moments calculated at the IP. Assuming the drift length between the collision point and the IP is L, the moments at collision point are:

$$\begin{aligned} < x^3 > =& < x_0^3 > +3 < x_0^2 p_{x0} > L + 3 < x_0 p_{x0}^2 > L^2 \\ & + < p_{x0}^3 > L^3 \\ < x^2 y > =& < x_0^2 y_0 > + < x_0^2 p_{y0} > L + 2 < x_0 p_{x0} y_0 > L \\ & + 2 < x_0 p_{x0} p_{y0} > L^2 + < p_{x0}^2 y_0 > L^2 \\ & + < p_{x0}^2 p_{y0} > L^3 \\ < x y^2 > =& < x_0 y_0^2 > +2 < x_0 y_0 p_{y0} > L + < x_0 p_{y0}^2 > L^2 \\ & + < p_{x0} y_0^2 > +2 < p_{x0} y_0 p_{y0} > L^2 \\ & + < p_{x0} p_{y0}^2 > L^3 \\ < y^3 > =& < y_0^3 > +3 < y_0^2 p_{y0} > L + 3 < y_0 p_{y0}^2 > L^2 \\ & + < p_{y0}^3 > L^3 \,, \end{aligned} \tag{20}$$

where the subscript "0" means the average is calculated at the IP.

In principle, we can extend this method to higher orders. However, the computation time increases significantly for higher orders. The truncation error is also intolerable. Therefore, we only preserve third-order moments in our code.

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Table 1: Flat Beam Parameters in the EIC CDR

Quantity	unit	proton	electron
Crossing angle	mrad	25	
Beam energy	GeV	275	10
Bunch intensity	10^{11}	0.668	1.72
β^* at IP	cm	80/7.2	55/5.6
Beam sizes at IP	μm	95/8.5	
Bunch length	cm	6	2
Transverse tunes		0.228/0.210	0.08/0.06
Longitudinal tune		0.01	0.069

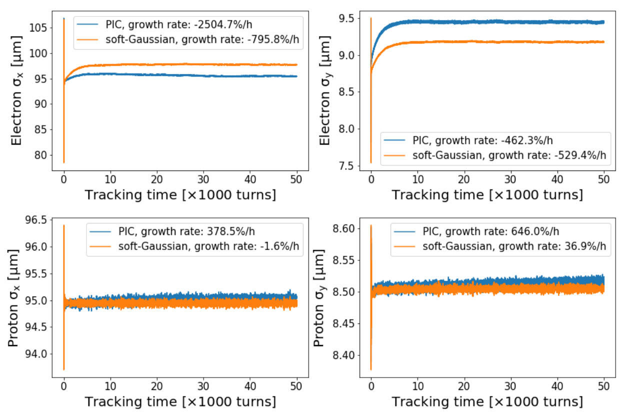

Figure 1: Strong-strong simulation by PIC code BeamBeam3D [6] and self-written soft-Gaussian code. The growth rate is linearly fitted from the last 60% tracking data.

APPLICATION TO EIC

Table 1 lists the beam parameters as presented in the EIC CDR [5]. Figure 1 compares the tracking results by PIC based strong-strong code BeamBeam3D [6] and self-written soft-Gaussian code. The equilibrium electron sizes are different in both codes because the soft-Gaussian is not self-consistent. Compared with BeamBeam3D, the soft-Gaussian code is less noisy. Figure 2 shows the proton size evolution by extended soft-Gaussian code. We can see that the 3rd order moments contribute to the horizontal and vertical size growth.

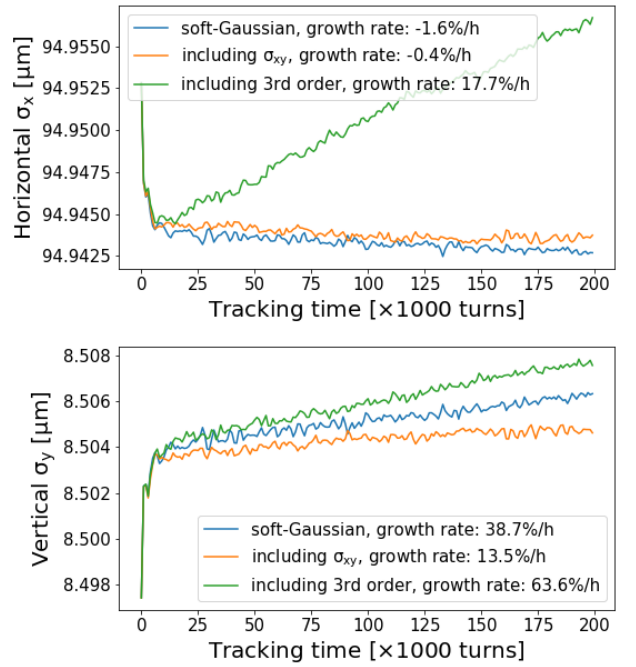

Figure 2: Tracking by extended soft-Gaussian code. The growth rate is linearly fitted from the last 60% tracking data. The data is averaged per 1000 turns to better show the growth trend.

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