# SPACE-CHARGE EFFECTS ON BUNCH ROTATION IN THE LONGITUDINAL PHASE SPACE

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*Abstract*

The longitudinal and transverse space-charge effects on bunch rotation in the longitudinal phase space designed to produce an intense short proton bunch are discussed. A criterion for length broadening due to space-charge modification of the rf potential is given. As for the transverse effect, the incoherent space-charge tune shifts will affect the bunch rotation unless the chromaticities are properly corrected.

## 1 INTRODUCTION

Intense proton bunches as short as 1 to 2 ns are necessary for the efficient production of pions and muons in order to (1) minimize the expensive cooling process of the muons and (2) to obtain a reasonable amount of polarization of $\mu^\pm$ in a muon collider and of $\nu_\mu$ and $\bar\nu_\mu$ in a neutrino factory. This can be done by bunch rotation in the longitudinal phase space prior to the extraction of the proton bunches. Here, we investigate the longitudinal and transverse space-charge effects on the intense beam during the rotation.

## 2 LONGITUDINAL EFFECTS

An experiment was performed at the IUCF Cooler Ring to study the rotation of proton bunches at 202.8 MeV below transition [1]. The ring has a circumference $C = 2\pi R = 86.83$ m, slip factor $\eta = -0.828$, and rf harmonic $h = 5$. The maximum rf voltage is $V_{\rm rf} = 1$ kV, corresponding to a small-amplitude synchrotron tune of $\nu_s = 1.164\times10^{-3}$. It was lowered adiabatically to a minimum of $\sim 5$ V and raised suddenly back to 1 kV. The bunch would be compressed in $\sim \frac{1}{4}$ synchrotron oscillation. The minimum bunch length observed was about $\sigma_\tau = 3.8$ ns.

The equations of motion governing the rf phase $\phi$ and fractional momentum spread $\delta$ in the bunch rotation are

$$\phi_{n+1} = \phi_n + 2\pi h|\eta|\delta_n \,, \qquad (1)$$

$$\delta_{n+1} = \delta_n + \frac{eV_{\rm rf}}{\beta^2 E}\sin\phi_{n+1} + \frac{\Delta U_{\rm spch}}{\beta^2 E} \,, \qquad (2)$$

where $n$ is the turn number, $E$ is the total particle energy, $\beta$ is the particle velocity relative to the velocity of light, and $e$ is the particle charge. The energy increase per turn due to the space-charge force is

$$\Delta U_{\rm spch} = -\frac{e^2}{\omega_0}\left|\frac{Z}{n}\right|_{\rm spch}\frac{\partial\rho}{\partial\tau} \,, \qquad (3)$$

where $f_0 = \omega_0/(2\pi)$ is the revolution frequency and $[Z/n]_{\rm spch}$ is the space-charge impedance. Because of the presence of electron cooling, we have assumed a Gaussian distribution for the longitudinal bunch profile $\rho(\tau) = N_b e^{-\tau^2/(2\sigma_\tau^2)}/(\sqrt{2\pi}\sigma_\tau)$, with $N_b = 1\times10^9$ is the number of particles in the bunch and $\sigma_\tau$ the rms bunch length.

The synchrotron tune of each particle will be reduced differently by the space-charge force, with the maximum at the core. This actually helps to reduce the nonlinearity of the rotation so as to let the tails of the bunch to catch up. The moment when the rms length is shortest will be delayed as the space-charge impedance increases. However, when the space-charge force is too large and exceeds the rf focusing force, the particles will embark on an unstable hyperbolic trajectory, and bunch lengthening results. Simulations for $|Z/n|_{\rm spch} = 2000$, 7000, and $15000\ \Omega$ are shown in Fig. 1 at the moment when the rms bunch length is shortest. As shown in Fig. 2, the minimum bunch length of $\sigma_\tau = 3.70$ ns is obtained when the impedance is at about $7000\ \Omega$.

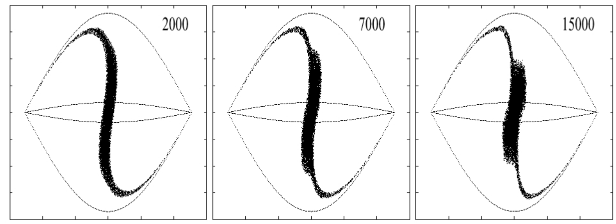


Figure 1: Phase-space plot of bunch rotation when the rms bunch length is shortest when $[Z/n]_{\rm spch} = 2000$, 7000, $15000\ \Omega$.

From Fig. 2, it appears that in order to have a final compressed bunch length $\sigma_\tau \lesssim 3.85$ ns, the space-charge impedance per harmonic must be limited to $|Z/n|_{\rm spch} \lesssim 15000\ \Omega$. In other words, the ratio of the space-charge force to the rf force must be less than the critical value of

$$\left.\frac{\text{Sp-ch force}}{\text{Rf force}}\right|_{\rm critical} = \frac{eN_b|Z/n|_{\rm spch}}{\sqrt{2\pi}h\omega_0^2\sigma_\tau^3 V_{\rm rf}} \sim 22.0 \,. \qquad (4)$$

It is important to point out that the actual space-charge impedance of the IUCF Cooler is only $|Z/n|_{\rm spch} \approx 1500\ \Omega$. What we are saying is that, while a rms bunch length $\sigma_\tau = 3.85$ ns can be obtained in the absence of space charge, a space-charge impedance as large as $|Z/n|_{\rm spch} \approx 15000\ \Omega$ will not lead to a longer compressed rms bunch length although the rf potential will be severely distorted.

The IUCF experiment is compared with the Fermilab proton driver in Table I. Notice that the space-charge-to-rf ratio for Phase II operation of the Fermilab proton driver is roughly at the critical value. Thus, we expect the bunch compression will not be affected longitudinally by the space-charge force.

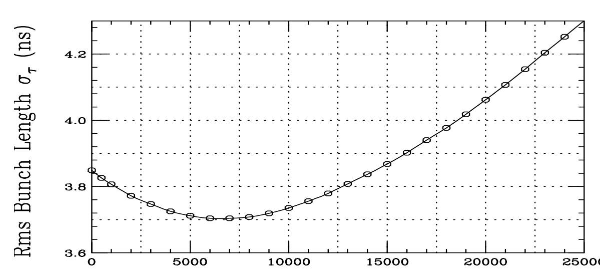


Figure 2: Plot showing shortest rms bunch length $\sigma_\tau$ obtained through rotation as a function of the space-charge impedance.

Table I: Comparison of the space-charge-to-rf ratio for IUCF experiment and Fermilab proton driver in Phases I and II.

| | IUCF Critical | Fermilab Proton Driver Phase I | Phase II |
|---|---|---|---|
| Circumference (m) | 86.83 | 711.32 | 711.32 |
| Extraction K.E. (GeV) | 0.203 | 16 | 16 |
| $h$ | 5 | 18 | 18 |
| No. per bunch $N_b$ | $1\,10^9$ | $1.7\,10^{12}$ | $2.5\,10^{13}$ |
| $\lvert Z/n\rvert_{\rm spch}$ ($\Omega$) | 15000 | 2.639 | 2.639 |
| $V_{\rm rf}$ (kV) | 1.00 | 1400 | 1400 |
| Extraction $\sigma_\tau$ (ns) | 3.85 | 3 | 1 |
| Sp-ch-to-rf ratio | 22.0 | 0.06 | 23.9 |

Our consideration so far emphasizes the effect of space-charge distortion of the rf potential. Microwave instability is less important during the rotation, because the local momentum spread increases. However, microwave instability can become a problem when the rf is lowered adiabatically so that the bunch fills the bucket. To avoid microwave instability, it will be better to employ instead the method of synchronous phase jump. The synchronous phase is jumped by $\pi$ so that the bunch center is at an unstable fixed point in the longitudinal phase space. The bunch will be lengthened along one set of separatrices and compressed along the other set. After some time $t$, the synchronous phase is jumped by $-\pi$ so that the bunch center is again at the stable fixed point. Allow for $\sim\frac{3}{8}$ of a synchrotron oscillation, the bunch will rotate to the situation of shortest length. This is illustrated in Fig. 3. The drift time $t$ along the separatrices cannot be too long. Otherwise, not all particles can return back to inside the bucket after the last phase switch. This allows us to derive the maximum possible compression ratio [1]

$$\left.\frac{(\sigma_\tau)_{\rm final}}{(\sigma_\tau)_{\rm initial}}\right|_{\rm max} \sim \frac{\sqrt{2}}{\sqrt{3}(\sigma_\phi)_i}\;, \tag{5}$$

where $(\sigma_\phi)_i$ is the initial rms bunch length in rf radian.

One can also avoid the development of nonlinear tails during the final rotation of $\sim\frac{3}{8}$ synchrotron oscillation. Instead of switching back to the stable fixed point, the bunch is extracted immediately at the end of the drift along the separatrices. The bunch is then sheared back to an upright position in the beam line via a lengthy optical system with local momentum compaction, or the $R_{56}$ element of the transfer matrix. Since this is not a rotation, the bunch length will be $\sqrt{2}$ longer than what was derived above. However, one may be able to recuperate this $\sqrt{2}$ by allowing the bunch to drift somewhat longer along the separatrices before the extraction. This is possible because the bunch need not be recaptured into a bucket later.

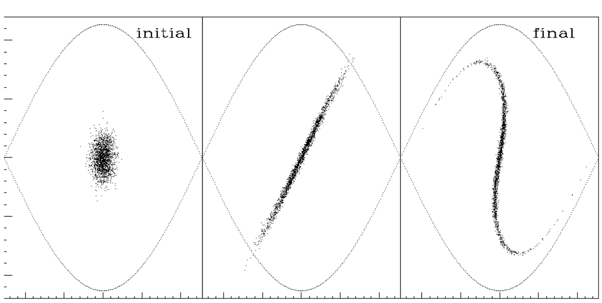


Figure 3: Bunch compression by rf phase jump. Note that the particle motion is relatively linear near the unstable fixed point.

## 3 TRANSVERSE EFFECTS

During bunch rotation, the bunch length is shortened and the incoherent space-charge tune shift increases. As an example, consider a former Fermilab design, which consists of a small ring with a circumference of 180.649 m accelerating protons from the K.E. of 1 GeV to 4.5 GeV. The rf harmonic is $h=4$ and there are $n_b=4$ bunches each with $N_b=5.0\times10^{13}$ protons. The 95% normalized emittance is $\epsilon_{\rm N95}=200\times10^{-6}$ $\pi$m. The incoherent space-charge tune shift at injection is

$$\Delta\nu_{sc} = -\frac{n_bN_br_p}{2\gamma^2\beta\epsilon_{\rm N95}B_f} = -0.131\;, \tag{6}$$

where the bunching factor $B_f=0.25$ has been used and a transverse uniform distribution has been assumed.

Bunch rotation is performed at the extraction K.E. of 4.5 GeV. When the rf voltage is reduced to its minimum, assume that the rms bunch length is $\sigma_\tau\sim\frac{1}{6}$ the bucket length. We would like to compress the bunch to $\sigma_\tau=1$ ns by suddenly raising the rf voltage. During bunch rotation, the bunching factor $B_f$ changes from 0.418 to 0.0164, while the incoherent space-charge tune shift changes from $\Delta\nu_{sc}=-0.009$ to $-0.225$ and different particles have different betatron tunes. If the reduction in betatron tune modifies the transition gamma to such an extent that some particles will find themselves near transition, higher order momentum compaction will be needed because of the large momentum spread. This may result in ruining the whole bunch rotation procedure as a result of nonlinearity.

### 3.1 A Theorem [2]

Let us first consider a storage ring without rf and neglect space charge. The canonical variables are $(a_x,J_x)$, $(a_y,J_y)$, and $(-\Delta\ell,\delta)$. Here $a_x$ and $a_y$ are the angle variables conjugate to the transverse actions $J_x$ and $J_y$. $\Delta\ell$ is the path length in excess of the length of the on-momentum closed orbit. The Hamiltonian will be cyclic in $a_x$, $a_y$, and $\Delta\ell$. It is given up to second order by

$$\begin{aligned}H &= \nu_{0x}J_x+\nu_{0y}J_y+\\ &\quad + aJ_x^2+2bJ_xJ_y+cJ_y^2+d\delta^2+fJ_x\delta+gJ_y\delta\;,\end{aligned} \tag{7}$$

with no $\mathcal{O}(\delta)$ term because comparison is made with respect to the on-momentum closed orbit. The betatron tunes are

$$\nu_x = \left\langle\frac{da_x}{d\theta}\right\rangle = \frac{\partial H}{\partial J_x} = \nu_{0x}+2aJ_x+2bJ_y+f\delta\;, \tag{8}$$

$$\nu_y = \left\langle\frac{da_y}{d\theta}\right\rangle = \frac{\partial H}{\partial J_y} = \nu_{0y}+2cJ_y+2bJ_x+g\delta\;, \tag{9}$$

and the path length difference per turn $\Delta\ell_0$ is

$$-\frac{\Delta\ell_0}{2\pi} = \left\langle-\frac{d\Delta\ell}{d\theta}\right\rangle = \frac{\partial H}{\partial\delta} = 2d\delta+fJ_x+gJ_y\;, \tag{10}$$

where $\langle\,\rangle$ denotes the average over one turn. We can readily identify $a$, $b$, and $c$ as amplitude-dependent detunings, $d=-\frac{1}{2}\alpha_0R$ with $\alpha_0$ the momentum compaction factor, and $f$

and $g$ as chromaticities. Thus, correcting the chromaticities will alleviate the dependence of path length on betatron amplitudes. The theorem can be extended by adding more higher order terms to the Hamiltonian, and some higher order chromaticity terms will enter into the right side of Eq. (10), which also require correction to avoid amplitude dependency on path length

### 3.2 Incoherent Tune Shift

Now let us add rf cavities. If the proton driver is dispersion-free at the cavities, the additional term in the Hamiltonian will not be dependent on the momentum deviation $\delta$, and therefore Eq. (10) will not be affected, even though this additional term in the Hamiltonian depends on the betatron oscillation amplitudes, $J_x$ and $J_y$.

We next include the self-field. Notice that the self-field space-charge tune shift in Eq. (6) is inversely proportional to $\gamma^3\beta^2$ (since $\epsilon_{\rm N95}\propto\gamma\beta$). Thus, the tune shift is momentum dependent and can be written as, with $z=x,y$,

$$\Delta\nu_z \approx \Delta\nu_{sc,z}\left(1-3\delta+12\delta^2\right) \ , \tag{11}$$

where $\Delta\nu_{sc,z}$ is evaluated at the nominal momentum. It is evident that the last two terms represent the first two lowest orders of chromaticity generated by the transverse space-charge force. Notice that the betatron action $J_z$ is related to the *unnormalized* emittance by $\epsilon = 2J_z$ and the transverse offset $z$ from the off-momentum closed orbit by $z=\sqrt{2\beta_zJ_z}$, where $\beta_z$ is the betatron function. For a round Kapchinskij-Vladimirskij (KV) beam [3] where the transverse distribution is uniform, $\Delta\nu_{sc,x}=\Delta\nu_{sc,y}=\Delta\nu_{sc}$ is $J_x$ and $J_y$ independent*. Thus, the contribution of the self-field space-charge tune shift to the Hamiltonian is

$$\Delta H=\Delta\nu_{sc}\left(J_x+J_y\right)\left(1-3\delta+12\delta^2\right)-\tfrac{1}{2}\Delta\alpha_{sc}R\delta^2 \ . \tag{12}$$

The first term gives the tune shifts and chromaticities provided by space charge. The last term is called Umstätter effect. It is the modification of the momentum compaction factor by space-charge tune shifts through the lattice. Although $\Delta\alpha_{sc}$ can be momentum dependent, it must be amplitude independent. If not, the space-charge tune shifts will be altered. For a FODO lattice, the change in transition gamma is roughly equal to the horizontal space-charge tune shift (exact for a uniform focusing lattice). For a flexible momentum compaction lattice, this term can be very much smaller. The additional chromaticities are

$$\Delta\xi_x=-3\Delta\nu_{sc}(1-8\delta)\,,\quad \Delta\xi_y=-3\Delta\nu_{sc}(1-8\delta)\ . \tag{13}$$

The additional changes in path length and $\gamma_T$ are

$$\frac{\Delta\ell_0}{C}=\Delta\nu_{sc}\frac{J_x+J_y}{R}(3-24\delta)-\left[\frac{2\Delta\nu_{sc}}{\gamma_T^3}\delta+\cdots\right]\,, \tag{14}$$

$$\Delta\gamma_T\approx 12\gamma_T^3\Delta\nu_{sc}\frac{J_x+J_y}{R}+\Delta\nu_{sc}\ . \tag{15}$$

In Phase II of the proton driver, the number per bunch is $N_b=2.5\times10^{13}$ and rf harmonic $h=18$. For the $\sigma_\tau=1$ ns compressed bunch, the bucket bunching factor is $B_f\approx\sqrt{2\pi}hf_0\sigma_\tau=0.01899$. With normalized 95% emittance $\epsilon_{\rm N95}=60\times10^{-6}\,\pi$m and an average betatron function of $\langle\beta_x\rangle=10$ m, the self-field space-charge tune shift is $\Delta\nu_{sc}=-0.297$ at extraction. The maximum actions for betatron motion are $J_x=J_y=1.67\times10^{-6}$ m. With the 2% momentum aperture in the vacuum chamber and the nominal transition gamma of $\gamma_T=j27.71$, the maximum contributions to the additional fractional path difference are $2.62\times10^{-8}$ for the first term of Eq. (14) and $5.57\times10^{-7}$ for the second. The maximum rf voltage used during the bunch rotation is $V_{\rm rf}=1.4$ MV, giving a synchrotron tune of $\nu_s=1.02\times10^{-3}$. Thus during the $\frac{1}{4}$-synchrotron-period bunch rotation, the total cumulative maximum additional path difference due to space-charge tune shift is $0.32\times10^{-6}$ for the first term and $4.56\times10^{-5}$ for the second term. On the other hand, the ratio of the rms bunch length at extraction to the ring circumference is $\sigma_\tau/T_0=42.11\times10^{-5}$, which is much larger, implying that the effect of space-charge tune shift on bunch compression through rotation is very minimal.

For the 4.5 GeV ring discussed earlier, with $\langle\beta_x\rangle\sim10$ m and $\sigma_\tau=1$ ns at extraction, $\Delta\nu_{sc}=-0.450$. With $V_{\rm rf}=4$ MV and $\gamma_T=j10$, the cumulative fractional path-length offset of $3.90\times10^{-4}$ is still much smaller than the ratio $\sigma_\tau/T_0=16.6\times10^{-4}$; the influence of space-charge tune shift is again small. On the other hand, the potential-well distortion due to space charge discussed in Sec. 2 can be more severe for this ring of the earlier design. For example, with a longitudinal space-charge impedance $|Z/n|_{\rm spch}=25\ \Omega$, the space-charge-to-rf ratio is as large as 47.3.

There has been the idea of changing the lattice near extraction so that the beam is near transition and the bunch narrowing effect near transition can be utilized [4]. The beam particles, however, will see a spread in space-charge tune shift of $\Delta\gamma_T\approx\Delta\nu_{sc}$ as a result of Umstätter effect. When the synchronous particle is less than $|\Delta\nu_{sc}|$ from the transition gamma, some particles will be above transition and some below making bunch rotation impossible.

It is important to point out that by having the $J_z\delta^2$ term in the additional Hamiltonian [Eq. (12)], we must include the same term into the original space-charge free Hamiltonian [Eq. (7)]. This is the next order chromaticity, which will contribute a down-shift to $\gamma_T$ just like the first term Eq. (15) with $\Delta\nu_{sc}$ replaced by $\frac{1}{24}(\xi_{x1}J_x+\xi_{y1}J_y)$, where $\xi_z=\xi_{z0}+\xi_{z1}\delta+\cdots$. For a linear machine, $\xi_{z1}=-2\xi_{z0}$. Thus, this order of chromaticity can lead to a much larger spread in $\gamma_T$ than the contribution from the space charge, and may require correction to ensure the bunch rotation.

*Even with other more realistic distributions, the result of the following discussions will not be much altered (see Ref. [5]).

## 4 REFERENCES

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