FOKKER-PLANCK SIMULATIONS OF BUNCHED BEAMS: HIGH-Q RF MODES AND RESPONSE FUNCTIONS

N. Towne*, National Synchrotron Light Source
Brookhaven National Laboratory, Bldg. 725B, Upton, NY 11973-5000†

Abstract

A computer code was developed to extend the method of Warnock and Ellison for integrating of the non-linear Vlasov-Fokker-Planck equation for bunched beams (The 2nd ICFA Advanced Accelerator Workshop on the Physics of High Brightness Beams, 1999). The code handles arbitrary radio-frequency potentials and high-Q impedances and is used to study instabilities in stretched bunches. This paper describes how high-Q resonant rf modes are incorporated into the code. A method by which this time-domain code is used to calculate beam response functions, which are response functions that include the beam acting back on itself through the ring impedance, is also described and applied to stretched bunches.

1 INTRODUCTION

Simulation of bunches in storage rings have been used to study the limiting behavior of instabilities. Most common is the simulation of instabilities driven by broadband impedance (short-range wakes) [1, 2, 3]. Simulation of instabilities driven by high-Q impedances is also done [4]. The simulations assume that the bunches are evolving in a harmonic radio-frequency (rf) potential. These studies have shed considerable light on the limiting of these instabilities and on relaxation phenomena. These studies have also highlighted the limits of linearized treatments of coherent modes and frequencies in bunched beams [5, 6, 7].

Two extensions of Warnock and Ellison's (W & E) [1] methods for the integration of the Vlasov-Fokker-Planck (VFP) equation were developed, one that permits simulation with non-harmonic rf potentials for Landau damping and lifetime improvement [8, 9] and a second that permits the inclusion of high-Q resonances with their long-range wakes, as well as a broad-band impedance, in the ring impedance. Although the former is a significant extension of W & E's methods and is essential for the study of the limiting of instabilities of stretched bunches [7], it is not discussed here [10] due to space limitations. The latter is discussed in Sec. 3. A third computational method that is not an extension of W & E's methods was also developed. It permits the calculation of frequency-domain beam response functions from time-domain simulations. Although the beam's response to the *total* voltage in the ring was treated by Shaposhnikova [11], this method for calculating the beam's response to an *externally applied* voltage, which does not include the voltage induced by the beam through the ring impedance and acting back upon the beam, is described and applied to a stretched bunch in Sec. 2.

2 BEAM RESPONSE FUNCTIONS

A bunch's longitudinal response to an external excitation is determined by its response to the total field in the ring, termed the beam transfer function (BTF) [11, 12], and by the field the bunch induces in the ring that in turn acts back on the bunch. The induced field arises from the ring's longitudinal impedance. This response to an external excitation is here termed the beam response function (BRF) and is useful for measuring the environment of the bunch, i.e., the longitudinal impedance.

Codes that simulate bunches in the time domain have the potential for providing a means to compute the frequency-domain BRF of a bunch. In a linear system, a frequency-domain response function is the Fourier transform of an appropriately defined impulse response. A time-domain code can readily calculate impulse responses since the impulse response is the evolution of the bunch from the appropriate initial condition at some starting time. So the problem is to determine the initial condition in the function space and the Fourier transform appropriate to determine the frequency-domain response function of interest—in this case the BRF. This is the task of this section.

Both the pickup and kicker are assumed to be located in the ring at azimuthal angle $\theta = 0$. We first consider the evolution of transients and impulse responses in stable bunches. One applies a voltage $V(\phi; t)$ to a bunch with the property that V goes to zero at $t = -\infty$. The co-moving coordinate ϕ is related to t and θ through $\theta = \omega_0 t + \phi$. The initial/boundary condition is that Ψ is the stationary Haïssinski distribution [13] Ψ_0 at $t = -\infty$. The bunch then evolves according to the linearized VFP equation giving a perturbed distribution

$$\delta\Psi(\phi, p; t) = \Psi(\phi, p; t) - \Psi_0(\phi, p; t) = B_{\text{VFP}}\left\{\delta(t) f(\phi)\right\}, \tag{1}$$

where B_{VFP} is the linear operator mapping functions $V(\phi; t)$ to phase space densities $\delta\Psi(\phi, p; t)$ via the VFP equation, and p is the momentum variable canonically conjugate to ϕ with respect to the rf Hamiltonian. The operator P generating the line density from the phase-space distribution by integrating away the variable p, and the Fourier transform $\mathcal{F}_m$ with respect to ϕ, are applied in turn to yield

* Email: towne@bnl.gov

† Work performed under the auspices of the U.S. Department of Energy, under contract DE-AC02-76CH00016.

a Fourier coefficient of the perturbed line density

$$\delta\rho_m(t) = \rho_m(t) - \rho_{0\,m}(t) = R_m\{V(\phi,t)\} \qquad (2)$$

where

$$R_m \equiv \mathcal{F}_m P B_{\mathrm{VFP}} \qquad (3)$$

and $\rho_{0\,m}$ is the mth Fourier coefficient of the Haïssinski line density. Now let the form of V be

$$V(\phi,t) = f(\phi)g(t) \qquad (4)$$

where f is an arbitrary phase dependence and $g(t) \to 0$ as $t \to -\infty$. R_m induces a map $H_{m\,f}$ of the functions g to line densities $\delta\rho_{m\,g}(t)$ through

$$H_{m\,f}\{g(t)\} \equiv \delta\rho_{m\,g}(t) = R_m\{f(\phi)g(t)\} \qquad (5)$$

where the f dependence of $\rho_{m\,g}$ is suppressed. Since the system is linear and time independent, $H_{m\,f}$ is a convolution.

$$\delta\rho_{m\,g}(t) = \int dt'\, H_{m\,f}(t-t')\, g(t') \qquad (6)$$

H is readily calculated for any given f with a time-domain simulation by specifying $g(t) = \delta(t)$ and simulating $H_{m\,f}(t) = \delta\rho_{m\,g}(t)$. The use of the delta function at $t = 0$ means that a non-zero initial condition at $t = 0$ that depends on f is specified for $\delta\Psi$ and that the integration begins at $t = 0$.

We now turn to the frequency domain. Frequency-domain response functions are ordinarily measured by exciting the bunch by a voltage $V(t) \propto e^{-i\omega t}$ with a steady sinusoidal time dependence at a frequency $\omega = \Omega + n\omega_0$ such that $|\Omega| \ll \omega_0$. Since ϕ is a co-moving coordinate such that $\theta = \omega_0 t + \phi = lT_0$ (l is an integer) where the pickup and kicker are located, this time dependence is

$$V = V(\phi;t) \simeq V_0 e^{-i(\Omega t - n\phi)} \qquad (7)$$

where V_0 is the peak voltage and $t = lT_0$ picks up the slow, and ϕ the fast, time dependence of the $V(t)$. Inserting Eq. (7) into Eq. (6), we get

$$\delta\rho_{m\,e^{-i\Omega t}}(t) = (C_{mn}(t) + iS_{mn}(t)) \circ e^{-i\Omega t} \qquad (8)$$

where the $\circ$ denotes convolution and the two functions C_{mn} and S_{mn} are defined

$$C_{mn}(t) \;=\; H_{m\,\cos n\phi}(t) \quad \text{and} \qquad (9)$$
$$S_{mn}(t) \;=\; H_{m\,\sin n\phi}(t) \qquad (10)$$

corresponding to the two terms of $e^{in\phi} = \cos n\phi + i \sin n\phi$. Each is calculated in a simulation providing $g(t) = \delta(t)$. Equation 8 is readily Fourier transformed to

$$\delta\tilde{\rho}_{m\,e^{-i\Omega t}}(\omega) = V_0\, T_{mn}(\omega)\, \delta(\omega - \Omega) \qquad (11)$$

where the beam response function $T_{mn}(\omega)$ is

$$T_{mn}(\omega) = (\tilde{C}_{mn}(\omega) + i\tilde{S}_{mn}(\omega))/2\pi \qquad (12)$$

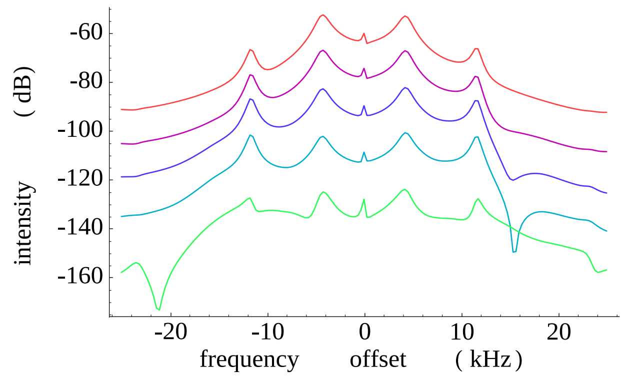

Figure 1: Simulated beam response functions $T_{9\,55}$, $T_{18\,55}$, $T_{36\,55}$, $T_{55\,55}$, and $T_{135\,55}$ (top to bottom) for a fully stretched 500-mA bunch in the National Synchrotron Light Source (NSLS) Vacuum Ultra-Violet (VUV) ring. Realistically detuned main- and harmonic-cavity impedances are included in the simulation. The vertical scale has an arbitrary scale factor. Machine parameters are given in Table 1.

and $\tilde{C}_{mn}$ and $\tilde{S}_{mn}$ are the Fourier transforms of C_{mn} and S_{mn}.

Equation Eq. (12) is the expression used to calculate beam response functions T_{mn} from a time-domain code. Two calculations are performed starting with stable bunches with Haïssinski distributions. One is given an initial kick with $\cos n\phi$ phase dependence, and the other with $\sin n\phi$ phase dependence. Each simulated line-density function of time is Fourier transformed with respect to ϕ at harmonic m and with respect to time to obtain $\tilde{C}_{mn}$ $\tilde{S}_{mn}$ and combined according to equation Eq. (12).

Figure 1 shows an example of a beam response function calculated using the method described in this section. The impedance of the ring includes the main- and harmonic-cavity accelerating-mode impedances with realistic detuning for beam-loading compensation. The peak with smaller offset is due to a dipole-like mode while the other peak is due to a quadrupole-like mode.

3 HIGH-Q RF MODES

Incorporation of high-Q rf modes in a time-domain simulation requires tracking the amplitude and phase of each mode with time as the modes and the bunch evolve in concert in the context of the VFP equation. Talman describes an rf mode as a two-dimensional real vector evolving according to a matrix equation describing the kicks the bunch imparts to each mode each turn [14]. The method described in this section uses a complex-valued quantity (phaser) $\tilde{V}_h$ representing the mode whose frequency ω_h is near the hth revolution line, i.e., $\omega_h - h\omega_0 \ll \omega_0$, where ω_0 is the revolution frequency. The real rf field in the mode is

$$V_h(t) = \mathrm{Re}[\tilde{V}_h(t)e^{i\omega_h t}] \qquad (13)$$

and $\tilde{V}_h(t)$ has slow time dependence. This section quickly derives the discrete-time evolution of $\tilde{V}_h(t)$.

The wake function for the rf mode is [15]

$$W_h(t) = \frac{k_h}{\cos\theta_h} e^{i(\Omega_h t + \theta_h)} + \text{c.c.} \tag{14}$$

where Γ_h is the damping rate, c.c. denotes the complex conjugate of the preceding term, $\Omega_h = \bar{\omega}_h + i\Gamma_h$, $k_h = \Gamma_h R_h$ is the loss factor, R_h is the impedance, $\bar{\omega}_h = \sqrt{\omega_h^2 - \Gamma_h^2}$, and

$$\tan\theta_h = \Gamma_h/\bar{\omega}_h \tag{15}$$

Let $z = e^{-i\Omega_h T_0}$. Then the recurrence relation for $\tilde{V}_h$ sampled turn by turn obtained by convolving the beam current expressed in terms of the line density and Eq. (14) is

$$\tilde{V}_h(lT_0) \simeq z^{-1}\tilde{V}_h((l-1)T_0) + k_h' \rho_h(lT_0) \tag{16}$$

where

$$k_h' = 2\pi \frac{k_h T_0 I_{\text{av}}}{\cos\theta_h} e^{i\theta_h} \tag{17}$$

$$\rho_h(lT_0) = \frac{1}{2\pi}\int_{2\pi} d\phi\, \rho(\phi, lT_0)\, e^{-ih\phi} \tag{18}$$

$T_0 = 2\pi/\omega_0$, and ρ is the line density with normalization $\int d\phi\, \rho = 1$. With the rf mode located in the ring at azimuthal angle $\theta = 0$ and the co-moving phase ϕ at turn l determined by $2\pi l = \omega_0 t + \phi$, the fast time dependence of $V_h(t)$ is provided by the ϕ dependence of the function

$$\begin{aligned} V_h(\phi, pT_0) &\equiv V_h(t) \\ &= \tilde{V}_h(pT_0)e^{-ih\phi} + \text{c.c.} \end{aligned} \tag{19}$$

Table 1: NSLS VUV ring and main- and harmonic-cavity parameters (separated by a forward slash), symbols, and values.

parameter	symbol	value
synchronous energy	E_0	800 MeV
energy loss per turn	U_0	20.4 keV
momentum compaction	α	0.0245
revolution frequency	ω_0	$2\pi \times 5.876$ MHz
radiation damping rate		1/7 ms
fractional energy spread	σ_ϵ	5×10^{-4}
rf harmonic numbers	h	9/36
rf peak voltages	V_h	80/19.7 kV
rf phases	ψ_h	$74.2°/-90°$
rf cavity impedances	R_h	435/100 kΩ
loaded quality factors	Q_h	6800/3360

4 CONCLUSION

An extension of the methods for the integration of the VFP equation developed by Warnock and Ellison [1], a code that tracks the particle distribution function in longitudinal phase space under the influence of a broad-band impedance, was described to permit high-Q impedances to be included in the ring impedance. A method for calculating beam response functions, which are response functions that include the beam-induced voltages acting back on the bunch, was also described and demonstrated with a stretched bunch.

5 ACKNOWLEDGMENTS

The author would like to thank Jiunn-Ming Wang and Sam Krinsky for many helpful discussions.

6 REFERENCES

[1] R. Warnock and J. Ellison, in *The 2nd ICFA Advanced Accelerator Workshop on the Physics of High Brightness Beams*, University of California, Los Angeles, November 9-12, 1999 (World Scientific, Singapore, 1999).

[2] B. V. Podobedov, *Saw-tooth Instability Studies at the Stanford Linear Collider Damping Rings*, Ph.D. thesis, Stanford University, Stanford Linear Accelerator Center, Stanford, CA 94309 (1999).

[3] M. Migliorati, L. Palumbo, G. Dattoli, and L. Mezi, Nucl. Instrum. Methods **A437**, 134 (1999).

[4] C. Limborg and J. Sebek, Phys. Rev. **E60**, 4823 (1999).

[5] T.-S. F. Wang, in *15th International Conference on High Energy Accelerators* (Hamburg, Germany, 1992).

[6] J. S. Berg and F. Ruggiero, in *17th IEEE Particle Accelerator Conference* (IEEE, 1997), invited talk.

[7] R. Bosch, *Robinson Instabilities with a Higher Harmonic Cavity* (2001), in preparation.

[8] Y. Miyahara, *Landau Damping of the Longitudinal Coupled-bunch Instability in an Electron Storage Ring*, Tech. Rep. ISSP 0082-4798, Institute for Solid State Physics, University of Tokyo (1983).

[9] R. Biscardi, S. L. Kramer, and G. Ramirez, Nucl. Instrum. Methods **A366**, 26 (1995).

[10] N. Towne, *Longitudinal Simulations and Measurements of Stretched Bunches* (2001), unpublished.

[11] E. N. Shaposhnikova, *Bunched Beam Transfer Matrices in Single and Double RF Systems*, Tech. Rep. SL-94-19-RF, CERN - SL Division, Geneva, Switzerland (1994).

[12] S. Krinsky and J. M. Wang, Part. Accel. **17**, 109 (1985).

[13] J. Haissinski, Nuovo Cimento **18B**(1), 72 (1973).

[14] R. Talman, *Z-Transform Description of a Bunch Acting on a Coupled Beam*, Tech. Rep. SSCL-Preprint-187, Superconducting Super Collider Laboratory, 2550 Beckleymeade Ave, Dallas, TX 75237 (1993).

[15] A. W. Chao, *Physics of Collective Beam Instabilities in High-energy Accelerators* (Wiley, New York, USA, 1993).