BEAM-FRAME CALCULATION OF FREE-ELECTRON LASER GAIN*

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Abstract

The amplification of radiation is calculated in the beam frame for a free electron laser (FEL) with a planar or helical undulator. The effect of the radiation force upon the transverse electron trajectory is included; this effect accounts for one-half of the gain in the undulator regime. Our calculated gain agrees with conventional formulas.

1 INTRODUCTION

In some FEL derivations, the effect of the radiation force upon the transverse electron motion ("force bunching" [1]) is neglected. When force bunching is neglected, using a standard method for calculating the axial velocity yields conventional gain formulas [1], while an alternative method yields one-half of the conventional gain [2]. Thus, it is inconsistent to assume that force bunching is negligible in an FEL [2].

We calculate FEL gain in the low-gain-per-pass undulator regime, performing our analysis in the frame of reference moving with the electron beam. In this frame, force bunching is easily included. We show that force bunching accounts for one-half of the bunching and gain in the undulator regime. For planar and helical undulators, our calculated gain agrees with conventional expressions, in which the helical gain is twice as large as the planar gain for a given wiggler parameter [3].

2 TRANSVERSE MOTION

For amplification of a weak radiation field by an ultrarelativistic beam, we include radiation in the transverse dynamics to model "force" bunching [1]. We first consider a planar undulator, in which an electron's velocity deviates by less than the angle $1/\beta\gamma$ from the z-axis, where $\gamma >> 1$ is the relativistic factor for the beam and $\beta \approx 1$ is the velocity divided by the speed of light c. In an undulator, the electron motion is non-relativistic in the frame of reference moving with the beam as it enters the undulator, so we calculate the dynamics in this frame using SI units. The relativistic factor and velocity describing this frame are denoted $\gamma_{\|}$ and $\beta_{\|}c$.

The undulator field appears in this frame as linearly polarized radiation traveling in the negative-z direction, with electric field in the x-direction

$$E_{wx}(z,t) = E_{wo}\cos(k_w z + \omega_w t) \tag{1}$$

The undulator magnetic field B_{wy} equals $-E_{wx}/c$, where $\omega_w = \beta_{\|}ck_w \approx ck_w > 0$. The undulator entrance obeys $k_w z+\omega_w t = 0$.

The radiation field is also linearly polarized, traveling in the z-direction, with

$$E_{rx}(z,t) = E_{ro}\cos(k_r z - \omega_r t + \phi_r) \tag{2}$$

and magnetic field $B_{ry} = E_{rx}/c$, where $\omega_r = ck_r > 0$.

Consider an electron with constant axial velocity v_z. The forced transverse oscillation from the undulator obeys

$$d^2x/dt^2 = (e/m)(1 + v_z/c)E_{wx}(z(t),t) \tag{3}$$

where $e < 0$ is the electron charge and m is its mass. The undulation velocity is therefore

$$v_{wx}(z,t) = -\hat{a}_w c\sin(k_w z + \omega_w t), \tag{4}$$

where

$$\hat{a}_w = -eE_{wo}/m\omega_w c \tag{5}$$

Similarly, the forced transverse oscillation velocity from the radiation is

$$v_{rx}(z,t) = a_r c\sin(k_r z - \omega_r t + \phi_r), \tag{6}$$

where

$$a_r = -eE_{ro}/m\omega_r c \tag{7}$$

Any axial velocity function may be approximated to arbitrary accuracy by constant-velocity segments, so that eqs. (4)–(7) also apply when the axial velocity is not constant. Since $v_{wx} = 0$ at the undulator entrance, a matched beam flows parallel to the axis with $\gamma_{\|} = \gamma$. Our assumption of non-relativistic velocities requires $\hat{a}_w << 1$.

3 AXIAL MOTION

To describe "inertial" bunching, we include radiation in the axial dynamics [1]. An electron whose initial axial position z is 0 obeys, to lowest order in the radiation field

$$\frac{d^2z}{dt^2} = \frac{e}{m}v_{wx}B_{wy} + \frac{e}{m}v_{wx}B_{ry} + \frac{e}{m}v_{rx}B_{wy} \ . \tag{8}$$

The solution with $z(0) = dz/dt(0) = 0$ is the sum of three functions describing radiation-independent axial motion, inertial bunching, and force bunching. The radiation-independent motion obeys $d^2z_o/dt^2 = (e/m)v_{wx}B_{wy}$ where $z \approx v_o t$ on the right hand side (RHS) of the equation, with v_o equaling the average axial velocity in the undulator. The solution with $z_o(0) = dz_o/dt(0) = 0$ is

$$z_o(t) = \frac{-e\hat{a}_w E_{wo}}{8m\hat{\omega}_w^2}(\sin 2\hat{\omega}_w t - 2\hat{\omega}_w t)\ , \tag{9}$$

where $\hat{\omega}_w \equiv \omega_w(1 + v_o/c)$ is the undulator frequency experienced by an electron with axial velocity v_o. For $\hat{a}_w << 1$, equation (9) gives the average axial velocity as

$$v_o = -\hat{a}_w^2 c/4 \tag{10}$$

Inertial bunching [1] results from the axial radiation force on an electron, obeying $d^2z_i/dt^2 = (e/m)v_{wx}B_{ry}$ where $z \approx z_o(t)$ on the RHS of the equation. For $\hat{a}_w << 1$, approximating $z_o(t) \approx v_o t$ on the RHS for the fundamental FEL mode gives the solution with $z_i(0) = dz_i/dt(0) = 0$

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$$z_{\mathrm{i}}(t) = \frac{e\hat{a}_w E_{ro}}{2m}\left[\begin{array}{l}\dfrac{\sin(\omega_+ t - \phi_r) + \sin\phi_r - \omega_+ t\cos\phi_r}{\omega_+^2} \\ + \dfrac{\sin(\omega_- t + \phi_r) - \sin\phi_r - \omega_- t\cos\phi_r}{\omega_-^2}\end{array}\right] \quad (11)$$

where $\omega_+ \equiv \hat{\omega}_w + \hat{\omega}_r$ and $\omega_- \equiv \hat{\omega}_w - \hat{\omega}_r$, in which $\hat{\omega}_r \equiv \omega_r(1 - v_o / c)$ is the radiation frequency experienced by an electron with axial velocity v_o. Since the undulation wavelength in the laboratory is independent of the electron's axial velocity, the inertial bunching is also called "axial" bunching [1].

Force bunching [1] results from the transverse radiation force on an electron, obeying $d^2 z_f / dt^2 = (e/m) v_{rx} B_{wy}$ where $z \approx z_o(t)$ on the RHS of the equation. For $\hat{a}_w << 1$, approximating $z_o(t) \approx v_o t$ on the RHS for the fundamental FEL mode gives the solution with $z_f(0) = dz_f/dt(0) = 0$

$$z_f(t) = \frac{-e a_r E_{wo}}{2m}\left\{\begin{array}{l}\dfrac{\sin(\omega_+ t - \phi_r) + \sin\phi_r - \omega_+ t\cos\phi_r}{\omega_+^2} \\ - \dfrac{\sin(\omega_- t + \phi_r) - \sin\phi_r - \omega_- t\cos\phi_r}{\omega_-^2}\end{array}\right\} \quad (12)$$

For effective amplification of radiation, $\omega_- << \omega_+$, so that

$$z_{\mathrm{i}}(t) = \frac{\omega_r}{\omega_w} z_f(t) = \frac{-a_r \hat{a}_w c\omega_r}{2}\left[\frac{\sin(\omega_- t + \phi_r) - \sin\phi_r - \omega_- t\cos\phi_r}{\omega_-^2}\right] \quad (13)$$

In the periodic undulator field, inertial bunching and force bunching are nearly equal when $\omega_- << \omega_+$.

4 GAIN

The change in an electron's energy from interaction with the radiation obeys

$$\frac{d\varepsilon}{dt} = e v_{rx} E_{rx} + e v_{wx} E_{rx} \quad (14)$$

where v_{rx}, v_{wx} and E_{rx} are evaluated at the axial position z(t) calculated in the previous section. The change in an average electron's energy is given by averaging over the phase of the radiation ϕ_r. To order E_{ro}^2, the first term on the RHS does not contribute to this average, so that for $\hat{a}_w^2 << 8$

$$\begin{aligned}\langle d\varepsilon/dt\rangle_{\phi_r} &= \langle e v_{wx} E_{rx}\rangle_{\phi_r} \approx \\ &\langle z\cos\phi_r\rangle_{\phi_r}\left[\frac{-e\hat{a}_w c E_{ro}}{2}(k_+\cos\omega_- t + k_-\cos\omega_+ t)\right] \\ &+ \langle z\sin\phi_r\rangle_{\phi_r}\left[\frac{e\hat{a}_w c E_{ro}}{2}(k_+\sin\omega_- t - k_-\sin\omega_+ t)\right]\end{aligned} \quad (15)$$

where $k_+ \equiv k_w + k_r$ and $k_- \equiv k_w - k_r$. Equation (13) gives

$$\begin{aligned}\langle z\cos\phi_r\rangle_{\phi_r} &= \left(1 + \frac{\omega_w}{\omega_r}\right)\frac{e\hat{a}_w E_{ro}}{4m\omega_-^2}(\sin\omega_- t - \omega_- t) \\ \langle z\sin\phi_r\rangle_{\phi_r} &= \left(1 + \frac{\omega_w}{\omega_r}\right)\frac{e\hat{a}_w E_{ro}}{4m\omega_-^2}(\cos\omega_- t - 1)\end{aligned} \quad (16)$$

where $1+\omega_w/\omega_r \approx 2/(1 - \hat{a}_w^2/4)$ for $\omega_- << \omega_+$.

Let $\Delta\varepsilon \equiv \int_0^T \langle d\varepsilon/dt\rangle_{\phi_r} dt$ be the average energy change per electron from interacting with radiation. Here, T is the undulator transit time, obeying $\hat{\omega}_w T = 2\pi N$ with integer or half-integer N equaling the number of undulator periods. Then, for $\omega_- << \omega_+$, eqs. (15) and (16) give

$$\Delta\varepsilon = -\frac{e^2 E_{ro}^2 \hat{a}_w^2 c k_+ T^3}{4m(1 - \hat{a}_w^2/4)}\left(\frac{2 - 2\cos\omega_- T - \omega_- T\sin\omega_- T}{\omega_-^3 T^3}\right). \quad (17)$$

In the beam frame, the number of electrons passing through the undulator within a transverse area A_o during a time t_o is $n_e A_o \beta c t_o$, so that the energy transferred to the forward wave is $-n_e A_o \beta c t_o \Delta\varepsilon$, where n_e is the electron density. The time-averaged Poynting vector of the radiation is $\langle S \rangle = \varepsilon_o c E_{ro}^2/2$, with energy density $\langle S \rangle/c$. Since the relative velocity between the forward wave and undulator is $(1+\beta)c \approx 2c$, the electromagnetic energy passing through the undulator is $(\langle S \rangle/c)(1+\beta)c t_o A_o$. The radiation energy gain per pass therefore obeys

$$\text{gain} = \frac{-n_e A_o \beta c t_o \Delta\varepsilon}{\langle S\rangle(1+\beta)\, t_o A_o} = \left(\frac{-2\beta}{1+\beta}\right)\frac{n_e \Delta\varepsilon}{\varepsilon_o E_{ro}^2}. \quad (18)$$

Equations (17) and (18) give for $\beta \approx 1$

$$\text{gain} = \frac{n_e e^2 k_+ c\hat{a}_w^2 T^3}{4m\varepsilon_o(1 - \hat{a}_w^2/4)}\left(\frac{2 - 2\cos\omega_- T - \omega_- T\sin\omega_- T}{\omega_-^3 T^3}\right). \quad (19)$$

In the laboratory frame, the maximum transverse velocity divided by c is obtained from the transverse and axial velocities in the beam frame when $|v_{wx}|$ is largest:

$$\beta_{\perp-lab} = a_w/\gamma \approx \hat{a}_w/[\gamma(1 - \hat{a}_w^2/2)], \quad (20)$$

where a_w is the wiggler parameter. The gain is given in the laboratory to lowest order in a_w:

$$\frac{n_{e-lab}\, e^2 \omega_{w-lab}\, a_w^2 L_{lab}^3}{2m\varepsilon_o c^3 \gamma^3}\left[\frac{2 - 2\cos\omega_- T - \omega_- T\sin\omega_- T}{(\omega_- T)^3}\right] \quad (21)$$

where n_{e-lab} is the e-beam density, $\omega_{w-lab} = ck_{w-lab}$ is the angular undulation frequency, and L_{lab} is the undulator length, all measured in the laboratory frame. Here,

$$\omega_- T = [\,k_{w-lab}(1 - a_w^2/4) - k_{r-lab}(1 + a_w^2/4)/2\gamma^2\,]\, cT_{lab} \quad (22)$$

where $T_{lab} = L_{lab}/c$ is the undulator transit time and k_{r-lab} is the radiation wave number in the laboratory. For optimal amplification, $\omega_- T = 2.61$, so that for $N >> 1$, $\gamma >> 1$ and $a_w << 1$, maximum gain occurs for

$$k_{r-lab} \approx 2\gamma^2 k_{w-lab}/(1 + a_w^2/2) \;. \quad (23)$$

For $a_w << 1$, the gain is twice as large as that resulting from inertial bunching alone.

5 HELICAL UNDULATOR

Consider a helical undulator in the frame of reference moving with a matched beam's axial velocity at the undulator entrance. Equations (1)–(7) are supplemented by equations describing radiation with y-polarization and motion in the y-direction. The additional undulator electric field is

$$E_{wy}(z,t) = E_{wo}\cos(k_w z + \omega_w t + \pi/2), \quad (24)$$

with additional magnetic undulator field $B_{wx} = E_{wy}/c$.

$$E_{ry}(z,t) = E_{ro}\cos(k_r z - \omega_r t + \phi_r - \pi/2) \quad (25)$$

describes the additional radiation whose magnetic field is $B_{rx} = -E_{ry}/c$. The undulation velocity in the y-direction is

$$v_{wy}(z,t) = -\hat{a}_w c \cos(k_w z + \omega_w t), \tag{26}$$

while the forced transverse oscillation velocity from the radiation in the y-direction is

$$v_{ry}(z,t) = -a_r c \cos(k_r z - \omega_r t + \phi_r), \tag{27}$$

Since $v_{wy} \neq 0$ at the helical undulator entrance (where $k_w z + \omega_w t = 0$), a matched beam has $\gamma_{\|} < \gamma$.

The axial motion of an electron whose initial axial position z is 0 obeys, to lowest order in the radiation field

$$\begin{aligned}\frac{d^2z}{dt^2} &= \frac{e}{m}\left(v_{wx}B_{wy} - v_{wy}B_{wx}\right) + \frac{e}{m}\left(v_{wx}B_{ry} - v_{wy}B_{rx}\right) \\ &+ \frac{e}{m}\left(v_{rx}B_{wy} - v_{ry}B_{wx}\right)\end{aligned} \tag{28}$$

The solution with initial conditions $z(0) = dz/dt(0) = 0$ is the sum of three functions describing radiation-independent axial motion, inertial bunching, and force bunching. The radiation-independent motion obeys $d^2z_o/dt^2 = (e/m)(v_{wx}B_{wy} - v_{wy}B_{wx})$ where $z \approx v_o t$ on the RHS, with v_o equaling the average axial velocity in the undulator. The solution with $z_o(0) = dz_o/dt(0) = 0$ is

$$z_o(t) = 0, \tag{29}$$

indicating that the average axial velocity v_o is zero.

The inertial bunching term obeys $d^2z_i/dt^2 = (e/m)(v_{wx}B_{ry} - v_{wy}B_{rx})$ where $z \approx 0$ on the RHS. The solution with $z_i(0) = dz_i/dt(0) = 0$ is

$$z_i(t) = \frac{e\hat{a}_w E_{ro}}{m\omega_-^2}\left[\sin(\omega_- t + \phi_r) - \sin\phi_r - \omega_- t\cos\phi_r\right] \tag{30}$$

where $\omega_- \equiv \omega_w - \omega_r$.

The force bunching term obeys $d^2z_f/dt^2 = (e/m)(v_{rx}B_{wy} - v_{ry}B_{wx})$ where $z \approx 0$ on the RHS. The solution with $z_f(0) = dz_f/dt(0) = 0$ is

$$z_f(t) = \frac{ea_r E_{wo}}{m\omega_-^2}\left[\sin(\omega_- t + \phi_r) - \sin\phi_r - \omega_- t\cos\phi_r\right]. \tag{31}$$

Thus,

$$z_i(t) = \frac{\omega_r}{\omega_w} z_f(t) = -a_r \hat{a}_w c\omega_r\left[\frac{\sin(\omega_- t + \phi_r) - \sin\phi_r - \omega_- t\cos\phi_r}{\omega_-^2}\right]. \tag{32}$$

Since $\omega_w/\omega_r = 1$ for $\omega_- << \omega_+ \equiv \omega_w + \omega_r$, force bunching accounts for one-half of the bunching.

To order E_{ro}^2,

$$\begin{aligned}\langle d\varepsilon/dt\rangle_{\phi_r} &= \langle ev_{wx}E_{rx} + ev_{wy}E_{ry}\rangle_{\phi_r} \\ &= -e\hat{a}_w cE_{ro}k_+\left(\cos\omega_- t\langle z\cos\phi_r\rangle_{\phi_r} - \sin\omega_- t\langle z\sin\phi_r\rangle_{\phi_r}\right)\end{aligned} \tag{33}$$

where $k_+ \equiv k_w + k_r$ and

$$\begin{aligned}\langle z\cos\phi_r\rangle_{\phi_r} &= \left(1 + \frac{\omega_w}{\omega_r}\right)\frac{e\hat{a}_w E_{ro}}{2m\omega_-^2}\left(\sin\omega_- t - \omega_- t\right) \\ \langle z\sin\phi_r\rangle_{\phi_r} &= \left(1 + \frac{\omega_w}{\omega_r}\right)\frac{e\hat{a}_w E_{ro}}{2m\omega_-^2}\left(\cos\omega_- t - 1\right)\end{aligned} \tag{34}$$

The average energy change per electron when $\omega_- << \omega_+$ is

$$\Delta\varepsilon = -\frac{e^2 E_{ro}^2 \hat{a}_w^2 ck_+ T^3}{m}\left(\frac{2 - 2\cos\omega_- T - \omega_- T\sin\omega_- T}{\omega_-^3 T^3}\right). \tag{35}$$

For a helical FEL, the time-averaged Poynting vector of the radiation is $\langle S\rangle = \varepsilon_o cE_{ro}^2$, with energy density $\langle S\rangle/c$. Since the relative velocity between the forward wave and undulator is $(1+\beta)c$, the electromagnetic energy passing through the undulator is $(\langle S\rangle/c)(1+\beta)ct_o A_o$. The radiation energy gain per pass therefore obeys

$$\text{gain} = \frac{-n_e A_o \beta ct_o \Delta\varepsilon}{\langle S\rangle(1+\beta)\,t_o A_o} = \left(\frac{-\beta}{1+\beta}\right)\frac{n_e \Delta\varepsilon}{\varepsilon_o E_{ro}^2}. \tag{36}$$

Equations (35) and (36) give for $\beta \approx 1$

$$\text{gain} = \frac{n_e e^2 k_+ c\hat{a}_w^2 T^3}{2m\varepsilon_o}\left(\frac{2 - 2\cos\omega_- T - \omega_- T\sin\omega_- T}{\omega_-^3 T^3}\right). \tag{37}$$

In the laboratory frame, the transverse velocity divided by c obeys

$$\beta_{\perp-lab} = a_w/\gamma = \hat{a}_w/\gamma_{\|} \tag{38}$$

where a_w is the wiggler parameter and γ is the relativistic factor for the beam. Using the relation $1/\gamma_{\|}^2 = (1+a_w^2)/\gamma^2$ we obtain the gain to lowest order in a_w

$$\frac{n_{e-lab}\, e^2 \omega_{w-lab} a_w^2 L_{lab}^3}{m\varepsilon_o c^3 \gamma^3}\left[\frac{2 - 2\cos\omega_- T - \omega_- T\sin\omega_- T}{(\omega_- T)^3}\right] \tag{39}$$

where

$$\omega_- T = [k_{w-lab} - k_{r-lab}(1 + a_w^2)/2\gamma^2]cT_{lab} \tag{40}$$

Maximum gain occurs for

$$k_{r-lab} \approx 2\gamma^2 k_{w-lab}/(1 + a_w^2). \tag{41}$$

The gain is twice as large as that from inertial bunching alone, and twice as large as that of a planar FEL with the same wiggler parameter. Our gain expression agrees with the conventional expression for a helical FEL [1].

6 SUMMARY

For planar and helical FELs, we have calculated the gain in the beam frame in the low-gain-per-pass undulator limit. Inertial and force bunching give equal contributions to the gain in the undulator regime. For planar and helical undulators, our calculated gain agrees with conventional expressions in which the helical gain is twice as large as the planar gain for a given wiggler parameter.

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