A THREE-DIMENSIONAL KINETIC THEORY OF CONTINUOUS-BEAM STABILITY *

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Abstract

This work is a three-dimensional stability study based on the modal analysis for a continuous beam with a Kapchinskij-Vladimirskij (KV) distribution. The analysis is carried out self-consistently within the context of linearized Vlasov-Maxwell equations and electrostatic approximation. The emphasis is on investigating the coupling between longitudinal and transverse perturbations in the high-intensity region. The interaction between the transverse modes supported by the KV distribution and those modes sustainable by the cold beam is examined. We found two classes of coupling modes that would not exist if the longitudinal and the transverse perturbations are treated separately. The effects of wall impedance on beam stability is also studied and numerical examples are presented.

INTRODUCTION

In a customary stability analysis of a continuous beam in an accelerator or storage ring, longitudinal and transverse effects are treated separately, an approximation that is valid because space-charge forces are relatively weak and characteristic frequencies differ by orders of magnitude. For a very intense beam like the one in the proposed heavy ion fusion facilities, the space-charge forces are large and all frequencies are of the order of the plasma frequency, the separated treatment of longitudinal and transverse perturbations may not be applicable. Such a concern was raised more than two decades ago in the heavy ion fusion studies. Since then, some investigations have been exploited in attempt to address the issue by improving the earlier stability theories for laminar beams or nearly laminar beams. In a study of two-dimensional, axisymmetric perturbations in a beam with a KV distribution, an instability caused by the coupling between the longitudinal and transverse motion was discovered in theory.[1] Later computer simulations confirmed the prediction and found this kind of instability to be a mechanism for energy exchange between the longitudinal and transverse motions in the beams with high anisotropy in temperature.[2-4] These findings and many fine papers published afterward[5-11] mark a success in exploring the intense beam stability. However, to date, the rigorous theory, though not necessarily computer simulations, is still left in the axisymmetric geometry and the three-dimensional theory remains to be improved. The purpose of this work is to extend the earlier investigation of axisymmetric modes in a KV beam to a full three-dimensional stability study. It is hoped that the approach and the results of this work will be helpful in the exploring and understanding of beam stability in non-axisymmetric geometry.

THEORETICAL MODEL

We consider a continuous, nonrelativistic beam of circular cross section with radius a and constant particle density ρ_0 propagating inside a conducting pipe of radius b and arbitrary wall impedance. A cylindrical coordinate system (r, φ, z) is chosen such that the beam is propagating in the positive z direction and the z axis coincides with the central axis of the beam. The equilibrium state of the beam is maintained by a constant linear external transverse focusing force which can be represented as $M\nu_0^2 r$ where M is the mass of a beam particle and ν_o is the betatron frequency in the absence of the beam's self-field. Taking the self-field of the beam into account, one finds the relation $\nu^2 = \nu_0^2 - (\omega_p^2/2)$, between the effective betatron frequency of particles ν, and the plasma frequency $\omega_p = (4\pi q^2\rho_0/M)^{1/2}$, where q is the charge of a beam particle. We assume the equilibrium distribution of beam particles in the phase space is described by the distribution function $f_0(\mathbf{x}, \mathbf{v})$ that is a product of the KV distribution in the transverse direction and a delta function of the longitudinal speed, i.e.

$$f_0(\mathbf{x}, \mathbf{v}) = \frac{\rho_0}{\pi}\delta\big[v_\perp^2 - \nu^2(a^2 - r^2)\big]\delta_z(v_z - v_o)\ , \quad (1)$$

where $v_\perp^2 = v_r^2 + v_\varphi^2$, v_r, v_φ and v_z are particles' radial, azimuthal and axial speeds, respectively, v_o is the averaged axial speed of particles, and $\delta(x)$ is the delta function.

STABILITY ANALYSIS

The stability study here is carried out within the context of the Vlasov-Maxwell equations and the electrostatic approximation for small perturbations evolving in the linear regime. Thus, we consider small perturbations in the distribution function $f_1(\mathbf{x}, \mathbf{v}, t)$ and in the electric potential $\phi_1(\mathbf{x}, t)$ described by the linearized Vlasov-Poisson equations

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{x}} + \frac{d\mathbf{v}}{dt}\cdot\frac{\partial f_1}{\partial \mathbf{v}} = \frac{q}{M}\nabla\phi_1\cdot\frac{\partial f_0}{\partial \mathbf{v}}\ , \quad (2)$$

and

$$\nabla^2\phi_1 = -4\pi q\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(\mathbf{x}, \mathbf{v}, t)d^3v\ . \quad (3)$$

Assuming the perturbed quantities vary in space and time according to $\{f_1, \phi_1\} = \{\tilde{f}, \tilde{\phi}\}e^{i(\omega t + m\varphi - kz)}$, the linearized Vlasov-Poisson equation can be treated by integrating over the unperturbed particle orbit to yield the following differential-integral equation in the region of $r \le a$,

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$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial\tilde{\phi}}{\partial r}\Big)-\Big(\frac{m^2}{r^2}+k^2\Big)\tilde{\phi}$$
$$=\frac{\omega_p^2}{a\nu^2}\tilde{\phi}(a)\delta(r-a)+\frac{2\omega_p^2}{\pi}\Big[i\Omega\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dv_r dv_\varphi\frac{d\delta_\perp}{dv_\perp^2}$$
$$\times\int_0^\infty\tilde{\phi}(r')\Big(\frac{1}{r'}\Big)^m\big(\xi_1 e^{i\theta}+\xi_2\big)^m e^{-i\Omega\tau}d\tau$$
$$+\frac{k^2}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dv_r dv_\varphi\delta_\perp\int_0^\infty\tau\tilde{\phi}(r')\Big(\frac{1}{r'}\Big)^m$$
$$\times\big(\xi_1 e^{i\theta}+\xi_2\big)^m e^{-i\Omega\tau}d\tau\Big]\,, \quad (4)$$

where $\delta_\perp=\delta\big[v_\perp^2-\nu^2(a^2-r^2)\big]$, $\xi_1=(v_\perp/\nu)\sin(\nu\tau)$, $\xi_2=r\cos(\nu\tau)$, $\Omega=\omega-kv_o$ is the Doppler-shifted frequency, $r'=(\xi_1^2+\xi_2^2+2v_r\xi_1\xi_2/v_\perp)^{1/2}$, and $m=0, 1, 2, \ldots$ denotes the azimuthal harmonic number. The right hand side of Eq. (4) is zero in the region of $a<r\le b$.

Expanding the perturbed electric potential in Eq. (4) as a sum of Jacobi polynomials $P_l^{(m,0)}(x)$ according to

$$\tilde{\phi}(r)=\Big(\frac{r}{a}\Big)^m\sum_{l=0}^{\infty}G_l P_l^{(m,0)}\Big(1-\frac{2r^2}{a^2}\Big)\,, \quad (5)$$

we can derive a recursion relation

$$W_l A_{l+1}+\big(W_l+W_{l-1}+U_l\big)A_l+W_{l-1}A_{l-1}=0\,, \quad (6)$$

for $l=1,2,3,\cdots$, where G_j is independent of r, $A_l=\sum_{j=l}^{\infty}(-1)^{j+l}G_j$,

$$U_l=2(m+2l)+(\omega_p/\nu)^2\big(B_{l-1}-B_l\big)\,, \quad (7)$$

$$W_l=\frac{a^2k^2}{2(m+2l+1)}\Big[1+\Big(\frac{\omega_p}{\nu}\Big)^2\frac{\partial B_l}{\partial\alpha}\Big]\,, \quad (8)$$

$$B_l=i\int_0^\infty e^{-i\alpha x}\cos^m x P_l^{(0,m)}(\cos 2x)dx\,, \quad (9)$$

and $\alpha=\Omega/\nu$. Applying the proper boundary conditions at $r=a$ together with Eq. (6) leads to the dispersion relation

$$\frac{a}{\tilde{\phi}_o}\frac{d\tilde{\phi}_o}{dr}\Big|_{r=a}=m+\Big(\frac{\omega_p}{\nu}\Big)^2\big(1-B_0\big)+W_0+\frac{W_0A_1}{A_0}\,, \quad (10)$$

where the ratio A_1/A_0 can be expressed in terms of infinite determinants or a continuous fractions, and

$$\tilde{\phi}_o(r)\sim I_m(kr)K_m(kb)-I_m(kb)K_m(kr)$$
$$-i\mathcal{Z}\big[I_m(kr)K_m'(kb)-K_m(kr)I_m'(kb)\big]\,, \quad (11)$$

is the potential external to the beam derive from solving Eq. (4) in the region of $a<r\le b$. Here, $I_n(x)$ and $K_n(x)$ are the nth order modified Bessel functions of the first and the second kinds, respectively, the prime indicates the derivative with respect to the argument, $\mathcal{Z}=\omega Z/(ck)$, Z is the wall impedance, and c is the speed of light.

For $k=0$, the recursion relation (6) reduces to the dispersion relation $U_j=0$ for the transverse modes discussed earlier in Ref. 12. When $m=0$, the Jacobi polynomials in Eq. (5) become Legendre polynomials and Eq. (10) reduces to the dispersion relation for axisymmetric modes studied in Ref. 1. Taking the limit of $\nu\to 0$ in Eq. (10), one finds the cold-beam dispersion relation[13]. The customary dispersion relation of the "usual dipole mode"[14] in a continuous nonrelativistic beam without axial momentum spread can be obtained from Eq. (10) by considering the limit of $kb\ll 1$ for $m=1$.

The roots of the dispersion relation (10) fall into three classes: (i) the ones that approach the pure transverse modes, i.e., the solutions of $U_j=0$, when $k\to 0$, (ii) the "high-frequency coupling modes" having the limit of $\Omega\to n\nu$ when $\omega_p\to 0$, and (iii) the "low-frequency coupling modes" with $\Omega\to 0$ when $\omega_p\to 0$. Both types of "coupling modes" are full three-dimensional perturbations and therefor vanish when $k=0$ or $m=0$ or when the longitudinal and the transverse perturbations are treated separately. The high-frequency coupling modes do not exist in the axisymmetric perturbations, and the low-frequency coupling modes exist only in the perturbations of even and zero m. The "usual transverse modes" found in the customary analyses[14] are similar to the lowest radial modes in class (i). When there is no strong necessity to distinguish the roots among the solutions of $U_j=0$, we shall use the notation $T_{m,j}$ to represent the whole family of solutions associated with $U_j=0$ for the mth azimuthal harmonic. The usual transverse modes will be referred to as the $T_{m,0}$ modes, the high-frequency coupling modes will be designated as $C_{m,j}$ modes, and the low-frequency coupling modes will be referred to as $L_{m,n}$ modes for $n\ge 1$, in the order of their first appearance in solving the dispersion relation using the $(2n-1)\times(2n-1)$ determinants.

NUMERICAL EXAMPLE

Here, we present a numerical example of the solutions to the dispersion relation (10) for some low radial modes associated with the dipole ($m=1$) perturbation. Readers are referred to Ref. 1 for the numerical results of the axisymmetric modes. We consider only the case of $b/a=1.5$ and $ka=1$. The infinite determinants in the dispersion relation have to be truncated to finite ranks for a practical numerical computation. We limit our study to the first sixteen transverse modes, up to the $T_{1,3}$ modes, out of the sequence of an infinite number of the roots of Eq. (10). The real part of Ω/ν_o is shown in Fig. 1 as a function of tune depression ν/ν_o. Figure 2 shows the real part of Ω^2/ν_o^2 as a function of ν/ν_o in the high-intensity region. As shown in Fig. 1, that for all modes, the values of Ω/ν_o start from the solutions of $U_j=0$ ($j=1,2,$ and 3), i.e. from 1, 3, 5, and 7, at $\nu=\nu_o$, and decrease when the beam intensity increases. When $\nu\to 0$, the $T_{1,0}$ mode approaches the cold-beam limit, while the Ω/ν_o of the upper $T_{1,2}$ and $T_{1,1}$ modes approach 2, and the Ω of all other modes approach zero.

A kind of obvious mode interaction appears in the high-intensity region as confluences of modes where two or more modes have the same real part of frequencies. Among the first sixteen roots, three confluences are found: the confluence of $T_{1,0}$ and $T_{1,2}$ near $\nu = 0.38\nu_o$, the confluence of $C_{1,1}$ and $T_{1,2}$ between $\nu = 0$ and $\nu = 0.26\nu_o$, and the confluence of two upper $T_{1,3}$s between $\nu = 0.118\nu_o$ and $\nu = 0.52\nu_o$. The frequencies in the confluence regions are complex conjugate pairs indicating possible instability. In addition, the lowest $T_{1,1}$ mode, the two lower $C_{1,2}$ modes, and the two upper $T_{1,3}$ modes are unstable in the high-intensity region. The two upper $T_{1,3}$s have the highest growth rate, about $0.72\nu_o$ at $\nu = 0.118\nu_o$ in the confluence and reaching $1.4\nu_o$ around $\nu \approx 0$. The lowest $T_{1,1}$ has the next highest growth rate of $0.088\nu_o$ near $\nu = 0.23\nu_o$. The confluence of $T_{1,0}$ and $T_{1,2}$ has a maximum growth rate of $0.001\nu_o$. We investigated the effect of resistive wall impedance and found that only the usual dipole mode, the $T_{1,0}$ mode, is appreciably influenced by the resistive wall impedance. The highest growth rate occurs near $\nu \approx 0$. In the case considered here, the maximal $|\mathrm{Im}(\Omega/\nu_o)|$ of the $T_{1,0}$ mode has the values of 0.0, 0.034, 0.066, and 0.092, for $\mathcal{Z} = 0.0$, 0.1, 0.2, and 0.3, respectively.

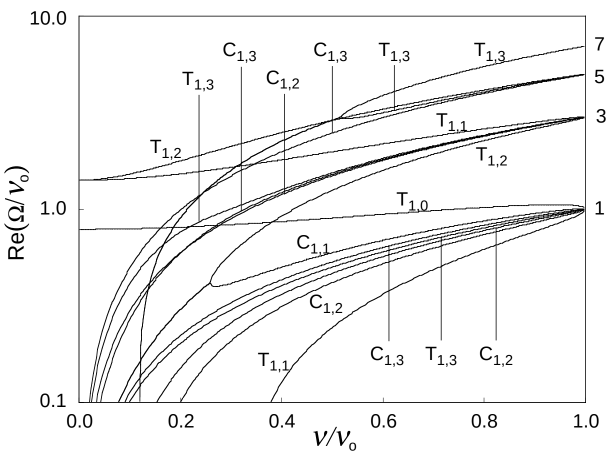

Figure 1: The real part of Ω/ν_o for the first sixteen $m = 1$ modes versus ν/ν_o for $ka = 1.0$, $b/a = 1.5$, and $Z = 0$.

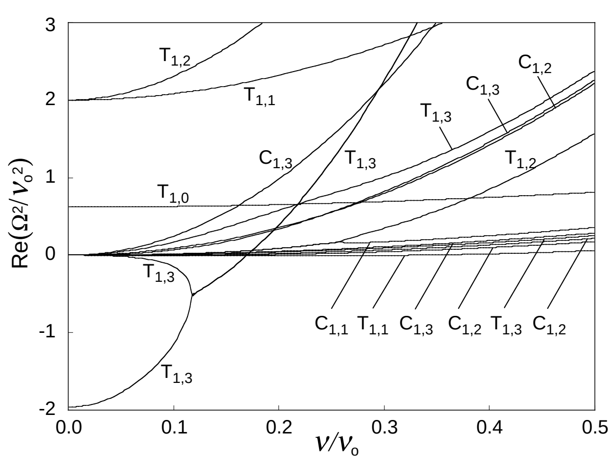

Figure 2: The real part of Ω^2/ν_o^2 for the first sixteen $m = 1$ modes as a function of ν/ν_o in the high-intensity region for $ka = 1.0$, $b/a = 1.5$, and $Z = 0$.

CONCLUSIONS

We have studied the three-dimensional stability of a continuous beam with a KV distribution within the context of linearized Vlasov-Maxwell equations and electrostatic approximation. A dispersion relation has been derived to facilitate the investigation of any azimuthal mode. Two classes of coupling modes were discovered. The occurrence of mode confluences in the high-intensity region indicates possible instability. We have examined some lower radial modes of dipole perturbation and identified some unstable modes. In particular, we have found a confluence of the usual dipole mode and a previously studied transverse mode may cause weak instability. The highest growth rate of the dipole modes are higher than that of the axisymmetric modes previously studied. Since not all instability in a KV beam are realized, computer simulations are suggested for further investigation. The effect of resistive wall impedance was also studied for dipole modes. It was found that only the usual dipole mode is appreciably affected by the resistive wall impedance.

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